Quantum Field Theory

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Outline

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Outline

Quantization of a One-Particle System

Operators

A particle whose dynamics is described by a conservative Hamiltonian ($\partial_t H = 0$): $H(q, p)$.

The quantum theory for this system is constructing by promoting position q and momentum p into operators:

$$q \longrightarrow \hat{q}$$

$$p \longrightarrow \hat{p}$$

with the commutation relation:

$$[\hat{q}, \hat{p}] = i\hbar$$

The Hamiltonian operator (with some *ordering prescription*) is

$$\widehat{H} = H(\hat{q}, \hat{p})$$

Operators act on states of a Hilbert space.

A basis $\{|q\rangle\}$ made of eigenstates of the position operator:

$$\hat{q}|q\rangle = q|q\rangle$$

$$\langle q'|q\rangle = \delta(q - q')$$

$$\int dq\, |q\rangle\langle q| = 1$$

Another basis $\{|n\rangle\}$ can be made out of eigenstates of the Hamiltonian (the energy states):

$$\widehat{H}|n\rangle = E_n|n\rangle$$

$$\langle n'|n\rangle = \delta_{nn'}$$

$$\sum_n |n\rangle\langle n| = 1$$

The ground state $|0\rangle$ is the state with the smallest energy E_0:

$$\widehat{H}|0\rangle = E_0|0\rangle$$

Schrödinger Picture

Time-independent operators:

$$\widehat{A}$$

Time-dependent states:

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}t\widehat{H}}|\psi\rangle$$

Heisenberg Picture

Time-dependent operators:

$$\widehat{A}(t) = e^{\frac{i}{\hbar}t\widehat{H}}\,\widehat{A}\,e^{-\frac{i}{\hbar}t\widehat{H}}$$

Time-independent states:

$$|\psi\rangle$$

The expectation value of $\widehat{A}$ in the state $|\psi\rangle$ after some time t in the two pictures:

$$\langle\psi(t)|\widehat{A}|\psi(t)\rangle = \langle\psi|\widehat{A}(t)|\psi\rangle$$

Position Expectation Values

A particle at time t_a is at position q_a:

$$|q_a\rangle$$

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The amplitude for finding the particle at q_b at t_b is:

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The amplitude for finding the particle at q_b at t_b is:

$$\langle q_b| e^{-i(t_b - t_a)\widehat{H}} |q_a\rangle$$

By dividing $(t_b - t_a)$ in N slices of size $\delta t = (t_b - t_a)/N$ and, at times $t_k = k\,\delta t$, inserting

$$\int dq_k \, |q_k\rangle\langle q_k|$$

it is possible to show ...

... that the amplitude is

$$\langle q_b | e^{-i(t_b - t_a)\widehat{H}} | q_a \rangle =$$

$$= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\prod_{i=1}^{N-1} \int dq_i \right) \left(\prod_{j=1}^{N} \int \frac{dp_j}{2\pi} \right) \exp \left[i\,\delta t \sum_{k=1}^{N} \left[p_k \frac{p_{k+1} - p_k}{\delta t} - H_k \right] \right]$$

$$\equiv \int_{\substack{q(t_a)=q_a \\ q(t_b)=q_b}} \mathcal{D}q(t) \int \mathcal{D}p(t) \exp \left[i \int_{t_a}^{t_b} \left[p(t)\dot{q}(t) - H(q,p) \right] dt \right]$$

where

$$H_k = H\left(\frac{q_k + q_{k-1}}{2}, p_k \right)$$

This expression defines the **path integral** for *this* problem.

Let us consider a system described by the Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

Then

$$H_k = \frac{p_k^2}{2m} + V_k$$

with

$$V_k = V\Big(\frac{q_k + q_{k-1}}{2}\Big)$$

Path Integral

Important Simple Example (2/2)

We can perform the integral over the momentum:

$$\langle q_b | e^{-i(t_b - t_a)\widehat{H}} | q_a \rangle =$$

$$= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\,2\pi\,\delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) \exp \left[i\,\delta t \sum_{k=1}^{N} \left[\frac{m}{2} \left(\frac{q_k - q_{k-1}}{\delta t} \right)^2 - V_k \right] \right]$$

$$= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\,2\pi\,\delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) \exp \left[i\,\delta t \sum_{k=1}^{N} L_k \right]$$

$$\equiv \int\limits_{\substack{q(t_a) = q_a \\ q(t_b) = q_b}} \mathcal{D}q(t)\, \exp \left[i \int_{t_a}^{t_b} L(q, \dot{q}) dt \right]$$

Path Integral

Important Simple Example (2/2)

We can perform the integral over the momentum:

$$\langle q_b | e^{-i(t_b - t_a)\widehat{H}} | q_a \rangle =$$

$$= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\,2\pi\,\delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) \exp\left[i\,\delta t \sum_{k=1}^{N} \left[\frac{m}{2} \left(\frac{q_k - q_{k-1}}{\delta t} \right)^2 - V_k \right] \right]$$

$$= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\,2\pi\,\delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) \exp\left[i\,\delta t \sum_{k=1}^{N} L_k \right]$$

$$\equiv \int_{\substack{q(t_a) = q_a \\ q(t_b) = q_b}} \mathcal{D}q(t)\, \exp\left[i \int_{t_a}^{t_b} L(q, \dot{q})\, dt \right]$$

Note the non-trivial constant: $(m/i\,2\pi\,\delta t)^{N/2}$.

Path Integral

Important Simple Example (2/2)

We can perform the integral over the momentum:

$$\langle q_b | e^{-i(t_b - t_a)\widehat{H}} | q_a \rangle =$$

$$= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\, 2\pi\, \delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) \exp\left[i\,\delta t \sum_{k=1}^{N} \left[\frac{m}{2} \left(\frac{q_k - q_{k-1}}{\delta t} \right)^2 - V_k \right] \right]$$

$$= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\, 2\pi\, \delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) \exp\left[i\,\delta t \sum_{k=1}^{N} L_k \right]$$

$$\equiv \int_{\substack{q(t_a) = q_a \\ q(t_b) = q_b}} \mathcal{D}q(t) \, \exp\left[i \int_{t_a}^{t_b} L(q, \dot{q}) dt \right]$$

Actually, one rarely has to compute a path integral.

Path Integral

Important Simple Example (2/2)

We can perform the integral over the momentum:

$$\langle q_b | e^{-i(t_b - t_a)\widehat{H}} | q_a \rangle =$$

$$= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\,2\pi\,\delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) \exp\left[i\,\delta t \sum_{k=1}^{N} \left[\frac{m}{2} \left(\frac{q_k - q_{k-1}}{\delta t} \right)^2 - V_k \right] \right]$$

$$= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\,2\pi\,\delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) \exp\left[i\,\delta t \sum_{k=1}^{N} L_k \right]$$

$$\equiv \int_{\substack{q(t_a) = q_a \\ q(t_b) = q_b}} \mathcal{D}q(t) \exp\left[i \int_{t_a}^{t_b} L(q, \dot{q}) dt \right]$$

To perform the complete calculation of a path integral, one would start from $\langle q_b | e^{-i(t_b - t_a)\widehat{H}} | q_a \rangle$.

Path Integral

Important Simple Example (2/2)

We can perform the integral over the momentum:

$$
\begin{aligned}
&\langle q_b | e^{-i(t_b - t_a)\widehat{H}} | q_a \rangle = \\
&= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\,2\pi\,\delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) \exp\left[i\,\delta t \sum_{k=1}^{N} \left[\frac{m}{2} \left(\frac{q_k - q_{k-1}}{\delta t} \right)^2 - V_k \right] \right] \\
&\qquad = \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\,2\pi\,\delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) \exp\left[i\,\delta t \sum_{k=1}^{N} L_k \right] \\
&\qquad\qquad \equiv \int_{\substack{q(t_a) = q_a \\ q(t_b) = q_b}} \mathcal{D}q(t) \exp\left[i \int_{t_a}^{t_b} L(q, \dot{q}) dt \right]
\end{aligned}
$$

One may use the Lagrangian form of the path integral; if L is not the actual Lagrangian, then one must use the Hamiltonian form.

From now on we will write

$$\boxed{\langle q_b | e^{-i(t_b - t_a)\widehat{H}} | q_a \rangle = \int\limits_{\substack{q(t_a)=q_a \\ q(t_b)=q_b}} \mathcal{D}q(t)\, \exp\left[i \int_{t_a}^{t_b} L(q, \dot{q})\, dt \right]}$$

without going into the details of the actual computation.

★ The free particle ($V(q) = 0$) is one of the few cases where we can find out an analytic expression for the amplitude $\langle q_b | e^{-i(t_b - t_a)\widehat{H}} | q_a \rangle$. The result is:

$$\langle q_b | e^{-\frac{i}{\hbar}(t_b - t_a)\widehat{H}} | q_a \rangle = \left[\frac{m}{i\,2\pi\hbar\,(t_b - t_a)} \right]^{1/2} \exp\left[\frac{i\,m\,(q_b - q_a)^2}{2\hbar\,(t_b - t_a)} \right]$$

★ For the harmonic oscillator:

$$L = \frac{1}{2}\,m\,\dot{q}^2(t) - \frac{1}{2}\,m\,\omega^2\,q^2(t)$$

$$\langle q_b | e^{-\frac{i}{\hbar}(t_b - t_a)\widehat{H}} | q_a \rangle =$$

$$= \left[\frac{m\,\omega}{i\,2\pi\hbar\,\sin[\omega(t_b - t_a)]} \right]^{1/2} \exp\left[\frac{i\,m\,\omega\,\left[(q_a^2 + q_b^2)\cos[\omega(t_b - t_a)] - 2q_a q_b \right]}{2\hbar\,\sin[\omega(t_b - t_a)]} \right]$$

In the $\omega \to 0$ limit, we recover result for a the free particle.

Ground-State Expectation Values
(1/2)

Most of the time we are interested in computing ground-state expectation values such as

$$\langle 0|e^{-i(t_b-t_a)\widehat{H}}|0\rangle$$

By inserting a complete set $\{|n\rangle\}$ of eigenstates of $\widehat{H}$ in $\langle q_b|e^{-i(t_b-t_a)\widehat{H}}|q_a\rangle$:

$$\begin{aligned}
\langle q_b|e^{-i(t_b-t_a)\widehat{H}}|q_a\rangle &= \sum_n \langle q_b|e^{-i(t_b-t_a)\widehat{H}}|n\rangle\langle n|q_a\rangle \\
&= \sum_n \langle q_b|n\rangle\langle n|q_a\rangle\, e^{-i(t_b-t_a)E_n}
\end{aligned}$$

Making $t_a = -T$ and $t_b = T$, in the $T \to \infty(1-i\epsilon)$ limit the dominant term is the one with the smallest E_n (the ground state). Therefore ...

Ground-State Expectation Values
(2/2)

...

$$\lim_{T\to\infty(1-i\epsilon)} \langle q_b | e^{-i(2T)\widehat{H}} | q_a\rangle = \langle q_b|0\rangle\langle 0|q_a\rangle \lim_{T\to\infty(1-i\epsilon)} e^{-i\,(2T)\,E_0}$$
$$= \langle q_b|0\rangle\langle 0|q_a\rangle \lim_{T\to\infty(1-i\epsilon)} \langle 0| e^{-i(2T)\widehat{H}} |0\rangle$$

and we conclude that

$$\lim_{T\to\infty(1-i\epsilon)} \langle 0| e^{-i(2T)\widehat{H}} |0\rangle = \frac{1}{\langle q_b|0\rangle\langle 0|q_a\rangle} \lim_{T\to\infty(1-i\epsilon)} \int_{\substack{q(t_a)=q_a \\ q(t_b)=q_b}} \mathcal{D}q(t)\, \exp\left[i \int_{-T}^{T} L(q,\dot{q})\, dt \right]$$

Since the left-hand side is independent of the initial al final positions, the dependence of the path integral on q_a and q_b must be cancel by the denominator. Therefore, no matter which q_a and q_b are used to perform the calculation, the result is the same.

Expectation Values of Time-Ordered Operators
(1/6)

The average position of a particle in the state $|\psi\rangle$ is $\langle\psi|\hat{q}|\psi\rangle$.

For a particle whose position is q_a at some initial time t_a and q_b at a later time t_b, what is the average position at a time t_r such that $t_b > t_r > t_a$? The answer:

$$\langle q_b|e^{-i(t_b-t_r)\widehat{H}}\,\hat{q}\,e^{-i(t_r-t_a)\widehat{H}}|q_a\rangle$$

Since (in the Heisenberg picture)

$$\hat{q}(t_1) = e^{i\,t_r\widehat{H}}\,\hat{q}\,e^{-i\,t_r\widehat{H}}$$

we can write

$$\langle q_b|e^{-i\,t_b\widehat{H}}\,\hat{q}(t_r)\,e^{i\,t_a\widehat{H}}|q_a\rangle$$

By inserting

$$\int dq_r|q_r\rangle\langle q_r|$$

at t_r, with $\hat{q}|q_r\rangle = q_r|q_r\rangle$, we have ...

Expectation Values of Time-Ordered Operators
(2/6)

... we have

$$\langle q_b | e^{-i(t_b - t_r)\widehat{H}}\, \hat{q}\, e^{-i(t_r - t_a)\widehat{H}} | q_a \rangle = \langle q_b | e^{-i\, t_b \widehat{H}}\, \hat{q}(t_r)\, e^{i\, t_a \widehat{H}} | q_a \rangle =$$

$$= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\, 2\pi\, \delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) q_r \exp \left[i\, \delta t \sum_{k=1}^{N} \left[\frac{m}{2} \left(\frac{q_k - q_{k-1}}{\delta t} \right)^2 - V_k \right] \right]$$

$$= \lim_{\substack{\delta t \to 0 \\ N \to \infty \\ N\,\delta t = t_b - t_a}} \left(\frac{m}{i\, 2\pi\, \delta t} \right)^{N/2} \left(\prod_{i=1}^{N-1} \int dq_i \right) q_r \exp \left[i\, \delta t \sum_{k=1}^{N} L_k \right]$$

$$\equiv \int_{\substack{q(t_a) = q_a \\ q(t_b) = q_b}} \mathcal{D}q(t)\, q(t_r)\, \exp \left[i \int_{t_a}^{t_b} L(q, \dot{q})\, dt \right]$$

Expectation Values of Time-Ordered Operators

(3/6)

On the other hand, by inserting two complete sets of eigenstates of $\widehat{H}$, setting $t_a = -T$ and $t_b = T$, and taking $T \to \infty(1 - i\epsilon)$:

$$\lim_{T\to\infty(1-i\epsilon)} \langle q_b | e^{-i\,T\widehat{H}}\, \hat{q}(t_r)\, e^{-i\,T\widehat{H}} | q_a \rangle = \langle 0 | \hat{q}(t_r) | 0 \rangle \langle q_b | 0 \rangle \langle 0 | q_a \rangle \lim_{T\to\infty(1-i\epsilon)} e^{-i\,(2T)\,E_0}$$

Since

$$\lim_{T\to\infty(1-i\epsilon)} \langle q_b | e^{-i(2T)\widehat{H}} | q_a \rangle = \langle q_b | 0 \rangle \langle 0 | q_a \rangle \lim_{T\to\infty(1-i\epsilon)} e^{-i\,(2T)\,E_0}$$

Dividing both expressions, we get

$$\langle 0 | \hat{q}(t_r) | 0 \rangle = \frac{\displaystyle\lim_{T\to\infty(1-i\epsilon)} \langle q_b | e^{-i\,T\widehat{H}}\, \hat{q}(t_r)\, e^{-i\,T\widehat{H}} | q_a \rangle}{\displaystyle\lim_{T\to\infty(1-i\epsilon)} \langle q_b | e^{-i(2T)\widehat{H}} | q_a \rangle}$$

$$= \frac{\displaystyle\lim_{T\to\infty(1-i\epsilon)} \int\limits_{\substack{q(t_a)=q_a\\ q(t_b)=q_b}} \mathcal{D}q(t)\, q(t_r)\, \exp\left[i \int_{-T}^{T} L(q,\dot{q})\, dt \right]}{\displaystyle\lim_{T\to\infty(1-i\epsilon)} \int\limits_{\substack{q(t_a)=q_a\\ q(t_b)=q_b}} \mathcal{D}q(t)\, \exp\left[i \int_{-T}^{T} L(q,\dot{q})\, dt \right]}$$

Expectation Values of Time-Ordered Operators

(4/6)

This result does not depend on the choice of q_a and q_b. This is a quite general feature, and it is therefore convenient to introduce a shorter notation for path integrals where q_a, q_b, and the $T \to \infty(1 - i\epsilon)$ limit are not shown explicitly:

$$\int \mathcal{D}q(t) \cdots \exp\left[i \int L(q, \dot{q})\, dt\right] \equiv$$

$$\lim_{T\to\infty(1-i\epsilon)} \int_{\substack{q(-T)=q_a \\ q(T)=q_b}} \mathcal{D}q(t) \cdots \exp\left[i \int_{-T}^{T} L(q, \dot{q})\, dt\right]$$

Also, we define the constant $\mathcal{N}$

$$\mathcal{N}^{-1} \equiv \int \mathcal{D}q(t) \exp\left[i \int L(q, \dot{q})\, dt\right]$$

so that we can write:

$$\boxed{\langle 0|\hat{q}(t_r)|0\rangle = \mathcal{N} \int \mathcal{D}q(t)\, q(t_r) \exp\left[i \int L(q, \dot{q})\, dt\right]}$$

Expectation Values of Time-Ordered Operators
(5/6)

Similarly, we can compute $\langle 0|\hat{q}(t_s)\hat{q}(t_r)|0\rangle$ with $t_b > t_s > t_r > t_a$ to get:

$$\langle 0|\hat{q}(t_s)\hat{q}(t_r)|0\rangle = \mathcal{N}\int \mathcal{D}q(t)\, q(t_s)\, q(t_r)\, \exp\left[i\int L(q,\dot{q})\, dt\right]$$

Now, if we compute $\langle 0|\hat{q}(t_r)\hat{q}(t_s)|0\rangle$ $t_b > t_r > t_s > t_a$, the result turns out to be the same:

$$\langle 0|\hat{q}(t_r)\hat{q}(t_s)|0\rangle = \mathcal{N}\int \mathcal{D}q(t)\, q(t_s)\, q(t_r)\, \exp\left[i\int L(q,\dot{q})\, dt\right]$$

These two results can be summarized as

$$\mathcal{N}\int \mathcal{D}q(t)\, q(t_s)\, q(t_r)\, \exp\left[i\int L(q,\dot{q})\, dt\right] = \begin{cases} \langle 0|\hat{q}(t_s)\hat{q}(t_r)|0\rangle & \text{for } t_s > t_r \\ \langle 0|\hat{q}(t_r)\hat{q}(t_s)|0\rangle & \text{for } t_s < t_r \end{cases}$$

Expectation Values of Time-Ordered Operators
(6/6)

★ It is conventional to define the time ordering operator $\mathcal{T}$ such that:

$$\mathcal{T}\{\hat{q}(t_s)\hat{q}(t_r)\} \equiv \begin{cases} \hat{q}(t_s)\hat{q}(t_r) & \text{for } t_s > t_r \\ \hat{q}(t_r)\hat{q}(t_s) & \text{for } t_s < t_r \end{cases}$$
$$= \theta(t_s - t_r)\,\hat{q}(t_s)\hat{q}(t_r) + \theta(t_r - t_s)\,\hat{q}(t_r)\hat{q}(t_s)$$

Then, we can write:

$$\langle 0|\mathcal{T}\{\hat{q}(t_s)\hat{q}(t_r)\}|0\rangle = \mathcal{N}\int \mathcal{D}q(t)\,q(t_s)\,q(t_r)\,\exp\left[i\int L(q,\dot{q})\,dt\right]$$

★ Similarly, for the product of n position operators:

$$\langle 0|\mathcal{T}\{\hat{q}(t_1)\cdots\hat{q}(t_n)\}|0\rangle = \mathcal{N}\int \mathcal{D}q(t)\,q(t_1)\cdots q(t_n)\,\exp\left[i\int L(q,\dot{q})\,dt\right]$$

★ In general:

$$\boxed{\langle 0|\mathcal{T}\{\hat{A}\}|0\rangle = \mathcal{N}\int \mathcal{D}q(t)\,A\,\exp\left[i\int L(q,\dot{q})\,dt\right]}$$

Quantization of an N-Particle System
(1/2)

The Lagrangian of a system with N particles is

$$L = \sum_{a=1}^{N} \frac{1}{2} m_a \dot{q}_a^2 - V(q_1, \ldots, q_N)$$

If we repeat our analysis, we will find expressions similar to those we obtained for one particle. For instance:

$$\begin{aligned}
\mathcal{N}^{-1} &= \int \mathcal{D}q(t)\, \exp\left[i \int L(q, \dot{q})\, dt \right] \\
&= \lim_{T \to \infty(1-i\epsilon)} \int\limits_{\substack{q_1(-T)=q_1^I, \cdots, q_N(-T)=q_N^I \\ q_1(T)=q_1^F, \cdots, q_N(T)=q_N^F}} \mathcal{D}q(t)\, \exp\left[i \int_{-T}^{T} L(q, \dot{q})\, dt \right]
\end{aligned}$$

also ...

Quantization of an N-Particle System
(2/2)

... also

$$\int\limits_{\substack{q_1(-T)=q_a^I,\cdots,q_N(-T)=q_N^I \\ q_1(T)=q_a^F,\cdots,q_N(T)=q_N^F}} \mathcal{D}q(t)\, \exp\left[i\int_{-T}^{T} L(q,\dot{q})\,dt\right] = \langle q_1^F\cdots q_N^F|e^{-i(2T)\widehat{H}}|q_1^I\cdots q_N^I\rangle$$

$$= \lim_{\substack{\delta t\to 0 \\ N_a\to\infty \\ N_a\,\delta t=2T}} \prod_{a=1}^{N}\left(\frac{m}{i\,2\pi\,\delta t}\right)^{N_a/2}\left(\prod_{i=1}^{N_a-1}\int d(q_a)_i\right)$$

$$\exp\left[i\,\delta t\sum_{a=1}^{N}\sum_{k=1}^{N_a}\left[\frac{m}{2}\left(\frac{(q_a)_k-(q_a)_{k-1}}{\delta t}\right)^2 - V_k\right]\right]$$

The lesson is that our results for one particle can be used for any number of particles as far as the path integral is interpreted correctly.

Outline

Canonical (Second) Quantization

(1/4)

★ We consider conservative systems (the Hamiltonian is independent of t) and take $\hbar = c = 1$.

★ Let us consider a relativistic field theory for a field $\varphi(x) = \varphi(t, \vec{x})$ described by the Lagrangian *density*

$$\begin{aligned}\mathcal{L}(\varphi, \partial_t\varphi) &= \frac{1}{2}\,(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}\,m^2\varphi^2 - V(\varphi)\\ &= \frac{1}{2}\,\dot{\varphi}^2 - \frac{1}{2}\,(\vec{\nabla}\varphi)^2 - \frac{1}{2}\,m^2\varphi^2 - V(\varphi)\end{aligned}$$

where I used $\dot{\varphi} \equiv \partial_t\varphi = \partial^t\varphi$. Although the actual Lagrangian is the volume integral of the Lagrangian density:

$$L = \int d^3x\,\mathcal{L}$$

it is very usual to say that $\mathcal{L}$ is *the Lagrangian*.

Canonical (Second) Quantization
(2/4)

★ The canonical conjugate field is:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \partial^t \varphi = \dot{\varphi}$$

The Hamiltonian *density* is:

$$\begin{aligned}\mathcal{H}(\varphi, \pi) &= \pi(x)\, \partial_t \varphi(x) - \mathcal{L} \\ &= \frac{1}{2}\, \pi^2 + \frac{1}{2}\, (\vec{\nabla}\varphi)^2 + \frac{1}{2}\, m^2 \varphi^2 + V(\varphi)\end{aligned}$$

and the Hamiltonian is:

$$H = \int d^3x\, \mathcal{H}$$

Canonical (Second) Quantization

(3/4)

★ The canonical quantization procedure, fields which are ordinary functions become operators:

$$\varphi(t,\vec{x}) \to \widehat{\varphi}(t,\vec{x})$$

The role of the momentum operator is played by

$$\pi(t,\vec{x}) \to \widehat{\pi}(t,\vec{x})$$

and $\widehat{\varphi}$ and $\widehat{\pi}$ satisfy the commutation relations:

$$\begin{aligned}
\left[\widehat{\varphi}(t,\vec{x}),\widehat{\pi}(t,\vec{x}\,')\right] &= i\,\delta(\vec{x}-\vec{x}\,') \\
\left[\widehat{\varphi}(t,\vec{x}),\widehat{\varphi}(t,\vec{x}\,')\right] &= 0 \\
\left[\widehat{\pi}(t,\vec{x}),\widehat{\pi}(t,\vec{x}\,')\right] &= 0
\end{aligned}$$

Note that since $\widehat{\varphi}(t,\vec{x})$ depends on t, it is a Heisenberg operator:

$$\widehat{\varphi}(t,\vec{x}) = e^{it\widehat{H}}\,\widehat{\varphi}(\vec{x})\,e^{-it\widehat{H}}$$

where

$$\widehat{H} = \int d^3x\,\mathcal{H}(\widehat{\varphi},\widehat{\pi})$$

Canonical (Second) Quantization

(4/4)

★ In our example:

$$\widehat{H} = \int d^3x \left[\frac{1}{2}\,\widehat{\pi}^2 + \frac{1}{2}\,(\vec{\nabla}\widehat{\varphi})^2 + \frac{1}{2}\,m^2\widehat{\varphi}^2 + V(\widehat{\varphi}) \right]$$

★ We can check out that:

$$\widehat{\pi} = \frac{d}{dt}\widehat{\varphi} = i\left[\widehat{H}, \widehat{\varphi}\right]$$

Second Quantization of the Schrödinger Equation
(1/7)

★ The Lagrangian

$$\mathcal{L} = \frac{i}{2}(\varphi^* \, \partial_t \varphi - \varphi \, \partial_t \varphi^*) - \frac{1}{2}(\partial_x \varphi^*)(\partial_x \varphi) - V(x) \, \varphi^* \varphi$$

describes a non-relativistic field theory in one dimension with two fields (φ and φ^*) whose Euler-Lagrange equations give the Schrödinger equation for the wave function of a particle in a one dimensional potential $V(x)$:

$$i \, \partial_t \varphi(t, x) = \left[-\frac{1}{2} \, \partial_x^2 + V(x) \right] \varphi(t, x)$$

The conjugate field is

$$\pi(t, x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} = i \, \varphi^*(t, x)$$

★ By second quantizing the Lagrangian, one obtains the (second quantization) Hamiltonian

$$\widehat{H} = \int dx\, \widehat{\varphi}^{\dagger}(t,x)\Big[-\frac{1}{2}\,\partial_x^2 + V(x)\Big]\,\widehat{\varphi}(t,x)$$

The commutation relations with $\widehat{\pi} = i\,\widehat{\varphi}^{\dagger}$ are

$$\begin{aligned}
\big[\widehat{\varphi}(t,\vec{x}), \widehat{\varphi}^{\dagger}(t,\vec{x}\,')\big] &= \delta(\vec{x}-\vec{x}\,') \\
\big[\widehat{\varphi}(t,\vec{x}), \widehat{\varphi}(t,\vec{x}\,')\big] &= 0 \\
\big[\widehat{\varphi}^{\dagger}(t,\vec{x}), \widehat{\varphi}^{\dagger}(t,\vec{x}\,')\big] &= 0
\end{aligned}$$

The equation

$$\frac{\partial\widehat{\varphi}}{\partial t} = i\,\big[\widehat{H}, \widehat{\varphi}\big]$$

says that $\widehat{\varphi}$ is a solution of the Schrödinger equation:

$$i\,\partial_t\widehat{\varphi}(t,x) = \Big[-\frac{1}{2}\,\partial_x^2 + V(x)\Big]\widehat{\varphi}(t,x)$$

Second Quantization of the Schrödinger Equation
(3/7)

★ Note that

$$\widehat{h} \equiv -\frac{1}{2}\,\partial_x^2 + V(x)$$

is the first quantization Hamiltonian whose eigenfunctions $\psi_n(t, x)$ and eigenvalues e_n are computed by using Quantum Mechanics methods:

$$\widehat{h}\,\psi_n(x) = e_n\,\psi_n(x)$$

Since the eigenfunctions of $\widehat{h}$ are a basis of the Hilbert space:

$$\int \psi_n^*(x)\,\psi_n(x)\,dx = \delta_{nm}$$

$$\sum_n \psi_n^*(x)\,\psi_n(x\,') = \delta(x - x\,')$$

then any solution of the Schrödinger equation can be written as a linear combinations of $\{\psi_n\}$.

Second Quantization of the Schrödinger Equation
(4/7)

★ Since $\widehat{\varphi}$ is a solution of the Schrödinger equation,

$$\widehat{\varphi}(t,x) = \sum_n \widehat{a}_n(t)\,\psi_n(x)$$

In this expression the expansion coefficients have to be operators because $\widehat{\varphi}$ is an operator.

The completeness relation for $\{\psi_n(x)\}$ and the commutations relations for $\widehat{\varphi}$ and $\widehat{\varphi}^\dagger$ give

$$\begin{aligned}
\left[\widehat{a}_n(t), \widehat{a}_m^\dagger(t)\right] &= \delta_{nm} \\
\left[\widehat{a}_n(t), \widehat{a}_m(t)\right] &= 0 \\
\left[\widehat{a}_n^\dagger(t), \widehat{a}_m^\dagger(t)\right] &= 0
\end{aligned}$$

Second Quantization of the Schrödinger Equation
(5/7)

★ Now we can write the (second quantization) Hamiltonian $\widehat{H}$ in terms of $\widehat{a}_n(t)$ and $\widehat{a}_n^\dagger(t)$ as

$$\widehat{H} = \sum_n e_n \, \widehat{a}_n^\dagger(t) \, \widehat{a}_n(t)$$

For n fixed, the operators $\widehat{a}_n(t)$ and $\widehat{a}_n^\dagger(t)$ are identical to the raising and lowering operators of the harmonic oscillator. The Hamiltonian is nothing but the sum of an infinite number of harmonic oscillator Hamiltonians. Following the discussion on the harmonic oscillator, we can develop a particle interpretation.

Second Quantization of the Schrödinger Equation
(6/7)

★ The lowest energy state of $\widehat{H}$, the ground state or bare vacuum, is the one that is empty. The destruction operator $\widehat{a}_n$ for any n finds no excitations (particles) to annihilate in the empty vacuum $|0\rangle$, so the result is the null vector:

$$\widehat{a}_n\,|0\rangle = 0$$

★ $\widehat{a}_n^\dagger|0\rangle$ is a state of energy e_n. It contains 1 particle of energy e_n created by $\widehat{a}_n^\dagger$.

★ $\widehat{a}_n^\dagger\widehat{a}_m^\dagger|0\rangle$ is a state of energy $e_n + e_m$. It is a 2-particle state created by $\widehat{a}_n^\dagger$ and $\widehat{a}_m^\dagger$.

★ The collection of all of the states spanned by the states formed by operating on $|0\rangle$ with any number of creation operators for any mode n is called a Fock space.

Second Quantization of the Schrödinger Equation

(7/7)

★ We have states and a particle interpretation for them: $\widehat{a}_n^\dagger(t)|0\rangle$ is a state at time t with 1 particle of energy e_n.

★ Since $\widehat{\varphi}(t,x)$ is expanded in terms of $\widehat{a}_n$ only and $\widehat{\varphi}^\dagger(t,x)$ is expanded in terms of $\widehat{a}_n^\dagger$ only, $\widehat{\varphi}(t,x)$ is a destruction operator and $\widehat{\varphi}^\dagger(t,x)$ is a creation operator:

$$\widehat{\varphi}^\dagger(t,x)|0\rangle$$

is a 1-particle state where the particle is located at position x at time t.

★ The quantization of the Schrödinger equation is special in some respects. It describes a non-relativistic theory and, therefore, the number of particles is fixed; actually, it can be shown that the Schrödinger equation for any fixed number of particles can be deduced from the quantum field theory.

Path Integrals for Quantum Field Theories

(1/4)

★ Let us consider a theory in one dimension for a field $\varphi(t,x)$ whose canonical conjugate field is $\pi(t,x)$ described by the Hamiltonian:

$$H = \int_0^L dx \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \varphi)^2 + \frac{1}{2}m^2\varphi^2 + V(\varphi) \right]$$

★ We assume a space region of length L that is a “lattice” of N points which are separated with each other by a distance l (eventually, we will take $l \to 0$ and $N \to \infty$ with L fixed).

★ Let us label each point by the letter a so that the values of fields φ and π at the point a are φ_a and π_a respectively. The Hamiltonian is, therefore,

$$H = \sum_{a=1}^{N} \frac{1}{2}\pi_a^2 + \sum_{a=1}^{N-1} \frac{1}{2}\left(\frac{\varphi_{a+1} - \varphi_a}{l}\right)^2 + \sum_{a=1}^{N} \frac{1}{2}m^2\varphi_a^2 + \sum_{a=1}^{N} V(\varphi_a)$$

which can be written as ...

Path Integrals for Quantum Field Theories

(2/4)

... which can be written as

$$H = \sum_{a=1}^{N} \frac{1}{2}\,\pi_a^2 + \sum_{a,b=1}^{N} h_{ab}\varphi_a\varphi_b + \sum_{a=1}^{N} V(\varphi_a)$$

This Hamiltonian describes a system of N particles which can be quantized using canonical quantization:

$$\begin{aligned} \varphi_a &\longrightarrow \widehat{\varphi}_a \\ \pi_a &\longrightarrow \widehat{\pi}_a \end{aligned}$$

with the canonical commutation relations:

$$\begin{aligned} \left[\widehat{\varphi}_a, \widehat{\pi}_b\right] &= i\,\delta_{ab} \\ \left[\widehat{\varphi}_a, \widehat{\varphi}_b\right] &= 0 \\ \left[\widehat{\pi}_a, \widehat{\pi}_b\right] &= 0 \end{aligned}$$

Path Integrals for Quantum Field Theories
(3/4)

★ The quantum Hamiltonian is:

$$\widehat{H} = \sum_{a=1}^{N} \frac{1}{2} \widehat{\pi}_a^2 + \sum_{a,b=1}^{N} h_{ab} \widehat{\varphi}_a \widehat{\varphi}_b + \sum_{a=1}^{N} V(\widehat{\varphi}_a)$$

Working out the path integral (as we did in QM), we can define:

$$\mathcal{N}^{-1}(L) \equiv \int \mathcal{D}\varphi \int \mathcal{D}\pi \, \exp\left[i \int \left[\pi_a \, \dot{\varphi}_a - H(\varphi, \pi) \right] dt \right]$$

Taking the limit $l \to 0$ and $N \to \infty$ with L fixed and performing the integration over π:

$$\mathcal{N}^{-1} = \int \mathcal{D}\varphi \, \exp\left[i \int \mathcal{L}(\varphi) \, dx \, dt \right]$$

The precise definition of the path integral is obtained by working out the expressions as in QM.

Path Integrals for Quantum Field Theories
(4/4)

This result can be generalized to any number of dimensions (again, the details about the path integral have to be worked out as in QM). In particular for a relativistic field theory:

$$\mathcal{N}^{-1} \equiv \int \mathcal{D}\varphi \, \exp\left[i \int \mathcal{L}(\varphi)\, d^4x \right]$$

★ The vacuum expectation value of a general time-ordered operator is given by the expression:

$$\boxed{\langle 0 | \mathcal{T}\{\widehat{A}\} | 0 \rangle = \mathcal{N} \int \mathcal{D}\varphi \, A \, \exp\left[i \int \mathcal{L}(\varphi) \right]}$$

★ The amplitudes for cross sections and decay rates are related to the *correlation functions* of the field:

$$\langle 0 | \mathcal{T}\{\widehat{\varphi}(x_1) \cdots \widehat{\varphi}(x_n)\} | 0 \rangle = \mathcal{N} \int \mathcal{D}\varphi \; \varphi(x_1) \cdots \varphi(x_n) \, \exp\left[i \int \mathcal{L}(\varphi) \right]$$

Cross Sections and Decay Rates

★ Golden rule for differential cross sections. For a process

$$A + B \to 1 + 2 + \cdots$$

the differential cross section is

$$d\sigma = |\mathcal{M}|^2 \frac{S}{\sqrt{(p_A \cdot p_B)^2 - (m_A\, m_B)^2}} \left(\prod_i \frac{d^3 p_i}{(2\pi)^3 2E_i} \right) (2\pi)^4\, \delta\Big(p_A + p_B - \sum_i p_i \Big)$$

★ Golden rule for differential decays. For a process

$$A \to 1 + 2 + \cdots$$

the decay rate is

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2m_A} \left(\prod_i \frac{d^3 p_i}{(2\pi)^3 2E_i} \right) (2\pi)^4\, \delta\Big(p_A - \sum_i p_i \Big)$$

★ In both expressions, if there are n_r identical particles of type r in the *final state*, the statistical factor S is

$$S = \prod_r \frac{1}{n_r!}$$

★ The amplitude $\mathcal{M}$ is obtained by using the Feynman rules with

$$i\,\mathcal{M} = \text{The sum of all connected, amputated diagrams.}$$

and on-shell external momenta: $p_a^2 = m_a^2$ for all $a = 1, 2, \ldots, n$ where n is the total number of particles involved in the process (incoming and outgoing particles). More precisely,

$$i\mathcal{M} = \left. \frac{Z^{n/2}\, G^{(n)}(p_1, \ldots, p_n)}{G^{(2)}(p_1, -p_1) \cdots G^{(2)}(p_n, -p_n)} \right|_{p_a^2 = m_a^2}$$

where

$$G^{(n)}(x_1, \ldots, x_n) = \langle 0 | \mathcal{T}\left\{ \widehat{\varphi}(x_1) \cdots \widehat{\varphi}(x_n) \right\} | 0 \rangle_{\text{conn}}$$

and Z is the field renormalization constant which we will find later. At this point, we do not worry about Z because, as we will see, we can give an expression for $i\mathcal{M}$ without explicitly mention Z.

★ In the previous result, connected means fully connected, that is, with no vacuum bubbles and all the external legs connected to each other.

Amputated means that the full propagator is removed from the external legs. This is accomplished by the propagators ($G^{(2)}$) in the denominator of the expression for $i\mathcal{M}$.

Green's Functions

(1/3)

★ Let us define a functional

$$\mathcal{Z}[J] \equiv \mathcal{N} \int \mathcal{D}\varphi \ \mathrm{exp} \left[i \int \mathcal{L}(\varphi) + J(x)\, \varphi(x) \right]$$

that depends on a function $J(x)$ called *the source*. By taking functional derivatives of $\mathcal{Z}$ with respect to $J(x)$, we get

$$\frac{1}{i^n} \frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1) \cdots \delta J(x_n)} = \mathcal{N} \int \mathcal{D}\varphi \, \varphi(x_1) \cdots \varphi(x_n) \ \mathrm{exp} \left[i \int \mathcal{L}(\varphi) + J(x)\, \varphi(x) \right]$$

If now we make $J = 0$, we obtain:

$$\begin{aligned} \left. \frac{1}{i^n} \frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} &= \mathcal{N} \int \mathcal{D}\varphi \, \varphi(x_1) \cdots \varphi(x_n) \ \mathrm{exp} \left[i \int \mathcal{L}(\varphi) \right] \\ &= \langle 0 | \mathcal{T} \left\{ \widehat{\varphi}(x_1) \cdots \widehat{\varphi}(x_n) \right\} | 0 \rangle \\ &= \mathcal{G}^{(n)}(x_1, \ldots, x_n) \end{aligned}$$

where $\mathcal{G}^{(n)}(x_1, \ldots, x_n)$ is a *Green's function*.

Green's Functions

(2/3)

★ Note that $\mathcal{Z}[J]$ is the generating functional of the Green's functions $\mathcal{G}^{(n)}(x_1, \ldots, x_n)$:

$$\mathcal{Z}[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{x_1} \cdots \int_{x_n} \mathcal{G}^{(n)}(x_1, \ldots, x_n)\, J(x_1) \cdots J(x_n)$$

★ Green's functions in momentum space are defined by

$$\widetilde{\mathcal{G}}^{(n)}(p_1, \ldots, p_n)\,(2\pi)^4 \delta(p_1 + \cdots + p_n) = \\ = \int_{x_1} \cdots \int_{x_n} \mathcal{G}^{(n)}(x_1, \ldots, x_n)\, e^{i(p_1 \cdot x_1 + \cdots + p_n \cdot x_n)}$$

Green's Functions

(3/3)

★ By the definition of $\mathcal{Z}[J]$, $\mathcal{Z}[0] = 1$.

★ Also, note that since $\mathcal{Z}[0] = \langle 0|0\rangle = 1$ which is consistent with a vacuum state that is normalized to one.

★ It is convenient to define

$$\mathcal{Z}[J] \equiv \langle 0|0\rangle_J$$

as the vacuum-vacuum amplitude in presence of a source $J(x)$.

★ Similarly,

$$\begin{aligned}\frac{1}{i^n}\frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1)\cdots\delta J(x_n)} &= \mathcal{N}\int \mathcal{D}\varphi\,\varphi(x_1)\cdots\varphi(x_n)\,\exp\left[i\int \mathcal{L}(\varphi) + J\,\varphi\right] \\ &\equiv \langle 0|\mathcal{T}\left\{\widehat{\varphi}(x_1)\cdots\widehat{\varphi}(x_n)\right\}|0\rangle_J\end{aligned}$$

is the correlation function in the presence of a source $J(x)$.

Connected Green's Functions

★ It is convenient to introduce a new functional:

$$i\, W[J] = \log \mathcal{Z}[J]$$

which is the generating functional of the connected Green's functions, $G^{(n)}$:

$$i\, W[J] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{x_1} \cdots \int_{x_n} G^{(n)}(x_1, \ldots, x_n)\, J(x_1) \cdots J(x_n)$$

$$G^{(n)}(x_1, \ldots, x_n) = \frac{1}{i^n} \frac{\delta^n (i\, W[J])}{\delta J(x_1) \cdots \delta J(x_n)} \bigg|_{J=0}$$

★ In momentum space:

$$\widetilde{G}^{(n)}(p_1, \ldots, p_n)\, (2\pi)^4 \delta(p_1 + \cdots + p_n) = \int_{x_1} \cdots \int_{x_n} G^{(n)}(x_1, \ldots, x_n)\, e^{i(p_1 \cdot x_1 + \cdots + p_n \cdot x_n)}$$

★ It is convenient to introduce the notation:

$$G^{(n)}(x_1, \ldots, x_n) = \langle 0 | \mathcal{T} \{ \widehat{\varphi}(x_1) \cdots \widehat{\varphi}(x_n) \} | 0 \rangle_{\text{conn}}$$

★ Note that $W[J = 0] = 0$ because $\mathcal{Z}[0] = 1$.

Free Field Theory

★ The Lagrangian for a free particle of mass m is

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}\, m^2 \varphi^2$$

The generating functional $\mathcal{Z}[J]$ is

$$\mathcal{Z}[J] = \mathcal{N} \int \mathcal{D}\varphi \;\exp\left[i \int \frac{1}{2}\,(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}\, m^2 \varphi^2 + J\,\varphi \right]$$

★ The integrand is oscillatory. To solve this problem we introduce a factor $e^{-\int \frac{1}{2}\epsilon\,\varphi^2}$. Eventually we will take $\epsilon \to 0$. The generating functional $\mathcal{Z}[J]$ is

$$\mathcal{Z}[J] = \mathcal{N} \int \mathcal{D}\varphi \;\exp\left[i \int_x \frac{1}{2}\,(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}\,(m^2 - i\,\epsilon)\,\varphi^2 + J\,\varphi \right]$$

Free Field Theory

★ Introducing the Fourier transforms for φ and J, we can write

$$\mathcal{Z}[J] = \mathcal{N} \int \mathcal{D}\phi \, \exp\left[\frac{i}{2}\int_k \widetilde{\phi}(k)(k^2-m^2+i\epsilon)^2\widetilde{\phi}(-k)\right] \exp\left[-\frac{i}{2}\int_k \frac{\widetilde{J}(k)\widetilde{J}(-k)}{k^2-m^2+i\epsilon}\right]$$

where we have defined

$$\widetilde{\phi}(k) \equiv \widetilde{\varphi}(k) + \frac{\widetilde{J}(k)}{k^2-m^2+i\epsilon}$$

★ The condition $\mathcal{Z}[J=0]=1$ gives

$$\mathcal{N}^{-1} = \int \mathcal{D}\phi \, \exp\left[\frac{i}{2}\int_k \widetilde{\phi}(k)(k^2-m^2+i\epsilon)^2\widetilde{\phi}(-k)\right]$$

and

$$\mathcal{Z}[J] = \exp\left[-\frac{i}{2}\int_k \frac{\widetilde{J}(k)\widetilde{J}(-k)}{k^2-m^2+i\epsilon}\right]$$

Free Field Theory

Trading $\widetilde{J}$ for J, we get

$$\mathcal{Z}[J] = \exp\left[-\frac{i}{2} \int_x \int_{x'} J(x)\, D(x-x')\, J(x') \right]$$

where

$$D(x-x') \equiv \int_k \frac{e^{-ik(x-x')}}{k^2 - m^2 + i\epsilon}$$

★ Note that the $i\epsilon$ prescription is essential; otherwise the k integral would hit a pole.

Free Field Theory

★ The 2-point Green function is

$$\mathcal{G}^{(2)}(x_1, x_2) = \langle 0|\mathcal{T}\{\widehat{\varphi}(x_1)\widehat{\varphi}(x_2)\}|0\rangle$$
$$= \frac{1}{i^2}\frac{\delta^2 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = i\,D(x_1 - x_2)$$

★ We define the Feynman propagator as

$$D_F(x_1 - x_2) \equiv i\,D(x_1 - x_2) = \int_k \frac{i\,e^{-ik(x_1 - x_2)}}{k^2 - m^2 + i\epsilon}$$

so that

$$\mathcal{G}^{(2)}(x_1, x_2) = \langle 0|\mathcal{T}\{\widehat{\varphi}(x_1)\widehat{\varphi}(x_2)\}|0\rangle = D_F(x_1 - x_2)$$

★ Physically, $D_F(x_1 - x_2)$ describes the amplitude for a disturbance in the field to propagate from x_1 to x_2 (or from x_2 to x_1 depending on the time order).

$\mathcal{G}^{(2)}(x_1, x_2) =$ x_1  x_2 $= D_F(x_1 - x_2)$

Free Field Theory

★ The 3-point Green function vanishes:

$$\mathcal{G}^{(3)}(x_1, x_2, x_3) = \frac{1}{i^3} \frac{\delta^3 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)}\bigg|_{J=0} = 0$$

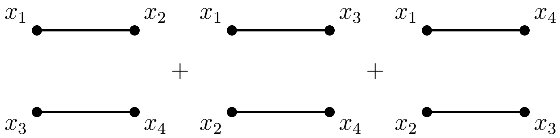
★ The 4-point Green function is

$$\mathcal{G}^{(4)}(x_1, x_2, x_3, x_4) = \frac{1}{i^4} \frac{\delta^4 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0} =$$

$$D_F(x_1 - x_2)\, D_F(x_3 - x_4) + D_F(x_1 - x_3)\, D_F(x_2 - x_4) + D_F(x_1 - x_4)\, D_F(x_2 - x_3)$$

Diagrammatically:

$\mathcal{G}^{(4)}(x_1, x_2, x_3, x_4) =$



Free Field Theory

★ Comments:

- Other functions can be computed in a similar way. However, it is more convenient to use diagrams like the ones we have just seen. These diagrams are called Feynman diagrams and with a set of simple rules can be used to compute any Green function: 1) draw n-points, 2) join them with lines (at most, one line per point) in all possible ways (each one gives a diagram), 3) if one point is left alone, the diagram vanishes, otherwise assign a Feynman propagator D_F to each line that joins pairs of points, and 4) summ all the possible diagrams.
- In general, for this theory, Green functions with an odd number of points (particles) vanish. The technical reason has to do with $\mathcal{Z}[J]$ being a function with 2 powers of J:

$$\mathcal{Z}[J] = \exp\left[-\frac{i}{2}\int_x\int_{x'} J(x)\, D(x-x')\, J(x')\right]$$

  Then, only by taking an even number of derivatives of $\mathcal{Z}$ we get terms independent of J which do not vanish at $J=0$.
- We also observe that Green's functions only depend on the coordinates difference. This reflects the fact that the theory is invariant under space-time translations.
- It is also interesting to note that Green's functions $\mathcal{G}^{(n)}$ with $n>2$ are disconnected.

Free Field Theory

★ In momentum space,

$$\widetilde{D}_F(k) = \frac{i}{k^2 - m^2 + i\epsilon}$$

One can also easily find out that

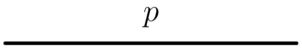
$$\widetilde{\mathcal{G}}^{(2)}(p_1, p_2)\,(2\pi)^4\delta(p_1 + p_2) = \widetilde{D}_F(p_1)\,(2\pi)^4\delta(p_1 + p_2)$$

Since for the 2-point Green function always $p_1 = -p_2$, we can write

$$\widetilde{\mathcal{G}}^{(2)}(p, -p) = \widetilde{D}_F(p)$$

Physically, $\widetilde{D}_F(p)$ describes the amplitude for a particle of mass m to propagate with momentum p. Diagrammatically,

$$\widetilde{\mathcal{G}}^{(2)}(p, -p) =$$



$$= \widetilde{D}_F(p)$$

Free Field Theory

★ The generating functional of the connected Green's functions is

$$i\,W[J] = \log \mathcal{Z}[J] = -\frac{i}{2}\int_x \int_{x'} J(x)\,D(x-x')\,J(x')$$

It is then clear that the connected Green functions are:

$$G^{(2)}(x_1, x_2) = D_F(x_1 - x_2)$$
$$G^{(n)}(x_1, \ldots, x_n) = 0 \text{ for } n > 2$$

Note that there is only one non-vanishing Green function and it is connected:

$$G^{(2)}(x_1, x_2) =$$

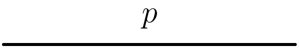
$$= D_F(x_1 - x_2)$$

In momentum space

$$\widetilde{G}^{(2)}(p, -p) = \widetilde{D}_F(p)$$

Diagrammatically

$$\widetilde{G}^{(2)}(p, -p) =$$



$$= \widetilde{D}_F(p)$$

Perturbation Theory

★ Let us write the Lagrangian $\mathcal{L}(\varphi)$ for a field theory as

$$\mathcal{L}(\varphi) = \mathcal{L}_0(\varphi) + \mathcal{L}_{\text{int}}(\varphi)$$

where $\mathcal{L}_0(\varphi)$ is the Lagrangian of the free theory and $\mathcal{L}_{\text{int}}(\varphi)$ describes the interaction. Then,

$$\begin{aligned}
\mathcal{Z}[J] &= \mathcal{N} \int \mathcal{D}\varphi \, \exp\left[i \int \mathcal{L}(\varphi) + J(x)\,\varphi(x) \right] \\
&= \mathcal{N} \exp\left[i \int \mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J} \right) \right] \int \mathcal{D}\varphi \, \exp\left[i \int \mathcal{L}_0(\varphi) + J(x)\,\varphi(x) \right] \\
&= \mathcal{N}' \exp\left[i \int \mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J} \right) \right] \mathcal{Z}_0[J]
\end{aligned}$$

where $\mathcal{Z}_0[J]$ is the generating functional of the free theory and

$$\mathcal{N}' = \mathcal{N}/\mathcal{N}_0$$

where $\mathcal{N}_0$ is the normalization constant of the free theory generating functional.

The φ^4 Theory

★ We will study a field theory described by the Lagrangian

$$\mathcal{L}(\varphi) = \mathcal{L}_0(\varphi) + \mathcal{L}_{\text{int}}(\varphi)$$

where $\mathcal{L}_0(\varphi)$ is the Lagrangian of the free theory:

$$\mathcal{L}_0 = \frac{1}{2}\,(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}\,m^2\varphi^2$$

and $\mathcal{L}_{\text{int}}(\varphi)$ describes the φ^4 interaction:

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\varphi^4$$

We know that for the free theory:

$$\mathcal{Z}_0[J] = \exp\left[-\frac{i}{2}\int_x\int_{x'} J(x)\,D(x-x')\,J(x')\right]$$

Therefore ...

The φ^4 Theory

... Therefore,

$$\mathcal{Z}[J] = \mathcal{N}\,' \exp\left[i \int \mathcal{L}_{\rm int}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right] \exp\left[-\frac{i}{2}\int_x \int_{x'} J(x)\, D(x-x')\, J(x')\right]$$

★ At order λ:

$$\begin{aligned}\mathcal{Z}[J] &= \mathcal{N}\,' \exp\left[i \int \mathcal{L}_{\rm int}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right] \mathcal{Z}_0[J] \\ &= \mathcal{N}\,' \left\{ \mathcal{Z}_0[J] - \frac{i\,\lambda}{4!}\int_x \frac{\delta^4 \mathcal{Z}_0[J]}{\delta J(x)^4} + \mathcal{O}(\lambda^2)\right\}\end{aligned}$$

By computing the functional derivatives, ...

The φ^4 Theory

..., we get

$$\mathcal{Z}[J] = \mathcal{N}'\,\mathcal{Z}_0[J]\left\{1 - \frac{i\,\lambda}{4!}\int_x \Big[3\,(iD_0)^2 - 6\,iD_0\,iD_{x1}\,iD_{x2}\,J_1\,J_2 + \right.$$
$$\left. + \, iD_{x1}\,iD_{x2}\,iD_{x3}\,iD_{x4}\,J_1\,J_2\,J_3\,J_4\Big] + \mathcal{O}(\lambda^2)\right\}$$

where

$$\begin{aligned} J_x &= J(x) \\ D_{xy} &= D(x-y) \\ D_0 &= D(0) = D(x-x) \end{aligned}$$

and repeated indices indicates integration over the corresponding variables.

The φ^4 Theory

★ We determine the constant $\mathcal{N}'$ by imposing $\mathcal{Z}[J=0]=1$. We get,

$$\mathcal{N}' = 1 + \frac{i\,\lambda}{4!}\int_x 3\,(iD_0)^2 + \mathcal{O}(\lambda^2)$$

Then

$$\mathcal{Z}[J] = \mathcal{Z}_0[J]\left\{1 - \frac{i\,\lambda}{4!}\int_x \Big[-6\, iD_0\, iD_{x1}\, iD_{x2}\, J_1\, J_2 + \right.$$

$$\left. + iD_{x1}\, iD_{x2}\, iD_{x3}\, iD_{x4}\, J_1\, J_2\, J_3\, J_4 \Big] + \mathcal{O}(\lambda^2)\right\}$$

The φ^4 Theory

★ The 2-point Green function is

$$\mathcal{G}^{(2)}(x_1, x_2) = \langle 0|\mathcal{T}\{\widehat{\varphi}(x_1)\widehat{\varphi}(x_2)\}|0\rangle = \frac{1}{i^2}\frac{\delta^2 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0}$$

$$= D_F(x_1 - x_2) - \frac{i\,\lambda}{2}\int_x D_F(x_1 - x)\, D_F(x - x)\, D_F(x - x_2) + \mathcal{O}(\lambda^2)$$

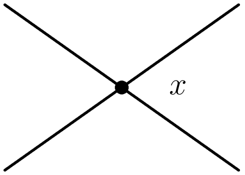
Now we can use diagrams to represent this result.

A propagator,

$D_F(x_1 - x_2)$

The φ^4 Theory

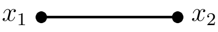
A “vertex”



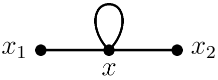
$$-i\,\lambda \int d^4x$$

Then we have

$\mathcal{G}^{(2)}(x_1, x_2) =$



$+\ \frac{1}{2}$



$+\ \mathcal{O}(\lambda^2)$

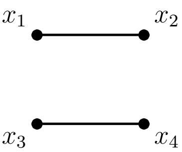
The φ^4 Theory

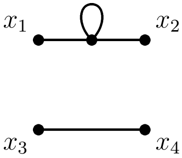
★ The 4-point Green function is

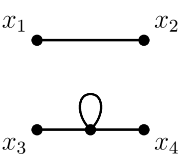
$$\mathcal{G}^{(4)}(x_1, x_2, x_3, x_4) = \frac{1}{i^4} \frac{\delta^4 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0} = \langle 0|\mathcal{T}\{\widehat{\varphi}(x_1)\widehat{\varphi}(x_2)\}|0\rangle$$

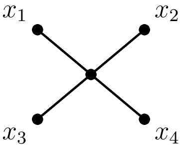
can be computed in a similar way. With diagrams:

$$\mathcal{G}^{(4)}(x_1, x_2, x_3, x_4) \; =$$

$=$ 

$+ \; \frac{1}{2}$ 

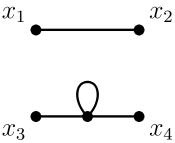
$+ \; \frac{1}{2}$ 

$+ \quad (2 \leftrightarrow 3) \quad + \quad (2 \leftrightarrow 4) \quad +$ 

$+ \quad \mathcal{O}(\lambda^2)$

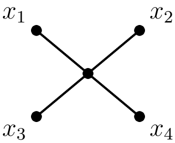
The φ^4 Theory

For instance,



$$= D_F(x_1 - x_2)\,(-i\lambda) \int d^4x\, D_F(x_3 - x)\, D_F(x - x)\, D_F(x - x_4)$$

and



$$= (-i\lambda) \int d^4x\, D_F(x_1 - x)\, D_F(x_2 - x)\, D_F(x_3 - x)\, D_F(x_4 - x)$$

The φ^4 Theory

★ The generating functional of the connected Green's functions is

$$i\,W[J] = \log \mathcal{Z}[J]$$

Since,

$$\mathcal{Z}[J] = \mathcal{Z}_0[J]\left\{1 - \frac{i\,\lambda}{4!}\int_x \Big[\cdots\Big] + \mathcal{O}(\lambda^2)\right\}$$

and

$$\mathcal{Z}_0[J] = \exp\left[-\frac{1}{2}\,iD_{12}\,J_1\,J_2\right]$$

we have,

$$i\,W[J] = -\frac{1}{2}\,iD_{12}\,J_1\,J_2 + \log\left\{1 - \frac{i\,\lambda}{4!}\int_x \Big[\cdots\Big] + \mathcal{O}(\lambda^2)\right\}$$

Using

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

we get

$$\begin{aligned} i\,W[J] &= -\frac{1}{2}\,iD_{12}\,J_1\,J_2 \\ &- \frac{i\,\lambda}{4!}\int_x \Big[-6\,iD_0\,iD_{x1}\,iD_{x2}\,J_1\,J_2 + iD_{x1}\,iD_{x2}\,iD_{x3}\,iD_{x4}\,J_1\,J_2\,J_3\,J_4\Big] + \mathcal{O}(\lambda^2) \end{aligned}$$

The φ^4 Theory

and

$$i\,W[J] = -\frac{1}{2}\left[iD_{12} + \frac{1}{2}(-i\,\lambda)\int_x iD_0\; iD_{x1}\; iD_{x2} + \mathcal{O}(\lambda^2)\right] J_1\, J_2$$
$$+\frac{1}{4!}\left[(-i\,\lambda)\int_x iD_{x1}\; iD_{x2}\; iD_{x3}\; iD_{x4} + \mathcal{O}(\lambda^2)\right] J_1\, J_2\, J_3\, J_4 + \mathcal{O}(\lambda^2)$$

Therefore,

$$G^{(2)}(x_1, x_2) = D_F(x_1 - x_2) +$$
$$+\frac{1}{2}(-i\lambda)\int_x D_F(x_1 - x)\, D_F(x - x)\, D_F(x_1 - x_2) + \mathcal{O}(\lambda^2)$$

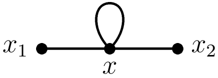
and

$$G^{(4)}(x_1, x_2, x_3, x_4) = (-i\lambda)\int_x D_F(x_1 - x)\, D_F(x_2 - x)\, D_F(x_3 - x)\, D_F(x_4 - x) + \mathcal{O}(\lambda^2)$$

The φ^4 Theory

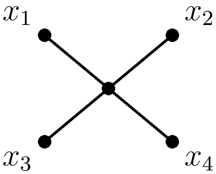
Diagrammatically,

$G^{(2)}(x_1, x_2) =$ 

$+ \frac{1}{2}$ 

$+ \mathcal{O}(\lambda^2)$

and

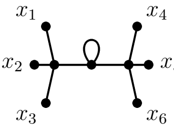
$G^{(4)}(x_1, x_2, x_3, x_4) =$ 

$+ \quad \mathcal{O}(\lambda^2)$

The φ^4 Theory

★ Comments.

- We observe that $i\,W[J]$ only generates connected Green functions.
- Any other Green function can be computed in a similar way. For instance,



$$= \frac{1}{2}\,(-i\lambda)^3 \int_x \int_y \int_z D_F(x_1 - y)\, D_F(x_2 - y)\, D_F(x_3 - y)$$
$$D_F(y - x)\, D_F(x - x)\, D_F(x - z)$$
$$D_F(z - x_4)\, D_F(z - x_5)\, D_F(z - x_6)$$

- The symmetry factor is in general the number of ways of interchanging components without changing the diagram. One rarely has to compute a symmetry factor larger that 2.

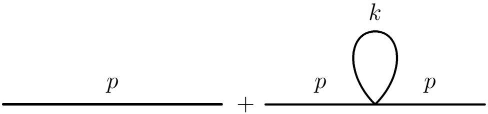
The φ^4 Theory

★ In momentum space, it is not difficult to find out

$$\widetilde{G}^{(2)}(p,-p) = \widetilde{D}_F(p) + (-i\,\lambda)\,\frac{1}{2}\,\widetilde{D}_F(p)\,\widetilde{D}_F(p)\,\int_k \widetilde{D}_F(k) + \mathcal{O}(\lambda^2)$$

which diagrammatically can be expressed as

$\widetilde{G}^{(2)}(p,-p) =$



$+\ \mathcal{O}(\lambda^2)$

This result tell us how to write the propagator of the full theory (with interaction) in terms of the free theory propagator.

Notation for the full propagator the full theory propagator:

$$\widetilde{D}(p) \equiv \widetilde{G}^{(2)}(p,-p)$$

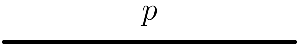
We can similarly define the full propagator in position space:

$$D(x_1 - x_2) \equiv G^{(2)}(x_1, x_2)$$

The φ^4 Theory

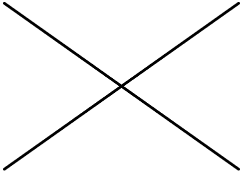
★ We can deduce the following Feynman rules:

1. Draw all possible diagrams.
2. Label each line with a momentum.
3. Momentum is conserved at each vertex.
4. Momenta associated with internal lines are to be integrated over with the measure
$$\int \frac{d^4 p}{(2\pi)^4}.$$
5. Find out the symmetry factor.
6. Propagator:



$$\frac{i}{p^2 - m^2 + i\,\epsilon}$$

7. Vertex:

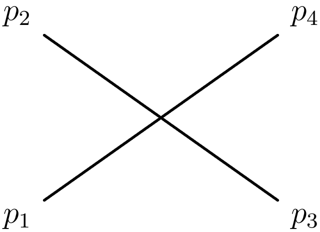


$$-i\,\lambda$$

The φ^4 Theory

★ Now we can use diagrams to compute other Green functions. For instance, the 4-particle connected Green function at order λ is

$$\widetilde{G}^{(4)}(p_1, p_2, p_3, p_4) \ =$$



$$+\ \mathcal{O}(\lambda^2)$$

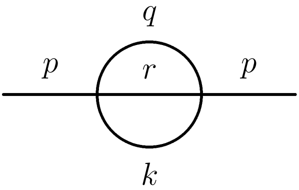
which gives

$$\widetilde{G}^{(4)}(p_1, p_2, p_3, p_4) = (-i\,\lambda)\,\frac{i}{p_1^2 - m^2 + i\,\epsilon}\,\frac{i}{p_2^2 - m^2 + i\,\epsilon}\,\frac{i}{p_3^2 - m^2 + i\,\epsilon}\,\frac{i}{p_4^2 - m^2 + i\,\epsilon}$$

with $p_1 + p_2 + p_3 + p_4 = 0$.

The φ^4 Theory

★ Another example. The following diagram contributes to $\widetilde{G}^{(2)}(p,-p)$ at order λ^2:



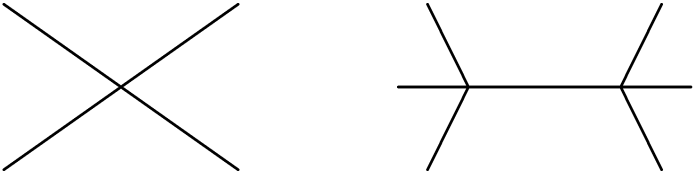
Note that $r = p + k + q$. We obtain,

$$\frac{1}{6}\,(-i\,\lambda)^2\left(\frac{i}{p^2 - m^2 + i\epsilon}\right)^2$$

$$\int_k \int_q \frac{i}{k^2 - m^2 + i\,\epsilon}\,\frac{i}{q^2 - m^2 + i\,\epsilon}\,\frac{i}{(p + k + q)^2 - m^2 + i\,\epsilon}$$

The φ^4 Theory

★ Diagrams without loops such as



are called tree diagrams. Calculations which are performed by considering only tree diagrams are tree-order calculations.

These calculations are important because it can be shown (by carefully inserting the factors of $\hbar$) that a diagram with L loops is of order $\hbar^L$. Therfore, a tree-order calculation is a calculation where the quantum corrections are ignored.

★ Notation Comments.

- From now on, we drop the tilde-symbol $\tilde{}$ for Green functions (and propagators) in momentum space. Usually it is clear from the context which kind of Green function we are referring to.
- Also, we will use an alternative notation for the free (Feynman) propagator D_F by defining

$$\Delta(p) \equiv D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}$$

1PI Diagrams and The Full Propagator

★ The diagrams that cannot be disconnected by cutting an internal line are called One Particle Irreducible (1PI) diagrams. These diagrams are special because any other diagram (connected or disconnected) can be constructed out of 1PI diagrams. This is a rather intuitive observation, but can be formulated in more precise terms.

★ Let us denote by $\Gamma^{(2)}_{1PI}$ the sum of all 1PI Feynman diagrams with 2 external lines. Then, the full propagator can written as

$$\begin{aligned} D &= \Delta + \Delta\,\Gamma^{(2)}_{1PI} + \Delta\,\Gamma^{(2)}_{1PI}\,\Delta\,\Gamma^{(2)}_{1PI} + \cdots \\ &= \frac{\Delta}{1 - \Gamma^{(2)}_{1PI}\,\Delta} \end{aligned}$$

Therefore,

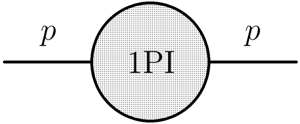
$$D^{-1} = \Delta^{-1} - \Gamma^{(2)}_{1PI}$$

This expression shows how to write the full propagator in terms of 1PI diagrams.

1PI Diagrams and The Full Propagator

★ For the φ^4 theory, we can write diagrammatically

$$D^{-1}(p) = -i\left(p^2 - m^2\right) -$$

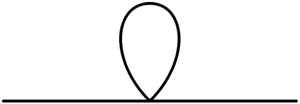


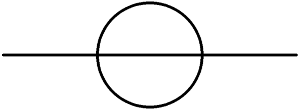
where the blob represents the sum of all 1PI Feynman diagrams with 2 external lines.

★ By construction, 1PI diagrams do not have propagators on the external lines.

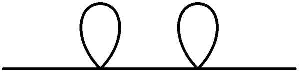
1PI Diagrams and The Full Propagator

These are 1PI diagrams:





But this diagram is *not* 1PI:

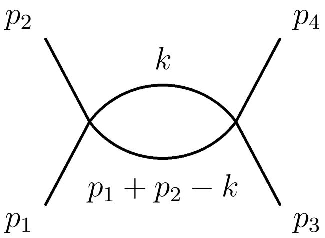


because it can be disconnected by cutting an internal line.

Outline

Divergences

★ Let us study the following diagram for the φ^4 theory:



contributes to $G^{(4)}$ at order λ^2. Applying the Feynman rules we obtain:

$$f(P^2) = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(P+k)^2 - m^2 + i\epsilon}$$

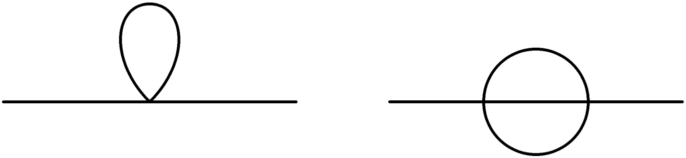
where $P = p_1 + p_2$. For large k, it goes like

$$f \sim \int^{\infty} \frac{k^3 dk}{k^4} \sim \int^{\infty} \frac{dk}{k}$$

which diverges logarithmically.

Divergences

★ It is pretty common to find out divergent Feynman diagrams. For instance, these diagrams are quadratically divergent:



For the first diagram we have 1 integration and 1 internal line which gives

$$\int^{\infty} \frac{k^3 dk}{k^2} \sim \int^{\infty} k\, dk$$

For the second diagram we have 2 integrals an 3 internal lines which give

$$\int^{\infty} \frac{k^7 dk}{(k^2)^3} \sim \int^{\infty} k\, dk$$

Divergences

★ In general, the (superficial) degree of divergence D is

$$D = 4\,L - 2\,I$$

where L is the number of integrals (loops) and I is the number of propagators (internal lines). For the φ^4 theory

$$4\,V = E + 2\,I$$

where E is the number of external lines.

★ A general (not only for φ^4) relation between the number of loops, internal lines, and vertices:

$$L = I - V + 1$$

Divergences

★ Using these expressions, the superficial degree of divergence:

$$D = 4 - E$$

For the φ^4 theory the superficial degree of divergence only depends on the number of external lines.

The 2-point diagrams are quadratically divergent ($D = 2$) and the 4-point diagrams are logarithmically divergent ($D = 0$). Diagrams with $D < 0$ are (superficially) convergent.

★ For obvious reasons, diagrams that diverge at large momentum (short distance) are called *ultraviolet* divergent diagrams.

Regularization

★ Ultraviolet divergences, apart from being a technical annoyance, have a profound physical meaning. When we compute a Feynman integral in the large momentum limit, we are assuming that our theory describes the short distance physics correctly.

Let us consider QED, the quantum theory of electromagnetism. We know that QED describes the physics of the atom so it is valid at distances of the order of atomic size; when we take the large momentum limit in Feynman integral we are saying that it also describes the interactions of charged particles at arbitrary small distances. However, we know that at short distance (roughly $\sim 1/M_W$) the weak interaction becomes important, even stronger than QED, and at even smaller distances, gravitation overpowers all the other interactions.

Regularization

★ The modern point of view is that quantum field theory is an effective low energy theory of a theory we do not yet know (string theory?) which should be valid up to some energy (momentum) scale Λ. Any physically sensible theory should have an implicit Λ.

Then, Feynman integrals with

$$\int \frac{d^4 p}{(2\pi)^4}$$

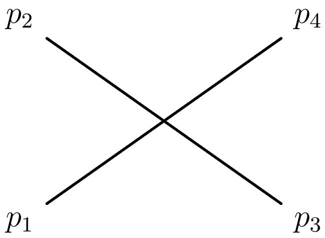
should be integrated only up to Λ, which is known as the cutoff. Then, we say that the integral has been "regularized".

★ Other regulators: Dimensional regularization, etc.

The Essence of Renormalization

★ Let us compute the 2-particle scattering amplitude of the φ^4 theory up to order λ^2. Two particles with momentum p_1 an p_2 collide producing two particles with momentum p_3 an p_4.

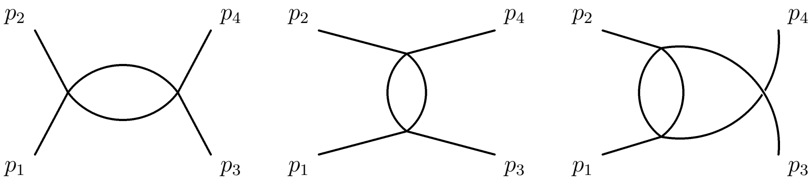
★ The tree order contribution is:



which simply gives $(-i\lambda)$.

The Essence of Renormalization

The one-loop diagrams:



are identical except for the external momenta and are given by

$$f(P^2) = \frac{1}{2}\,(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2 - m^2 + i\epsilon}\,\frac{i}{(P+k)^2 - m^2 + i\epsilon}$$

with $P = p_1 + p_2$, $P = p_1 - p_3$, and $P = p_1 - p_4$, respectively. Using an ultraviolet cutoff, we can show that

$$f(P^2) = i\,\lambda^2\,\frac{1}{32\pi^2}\,\log\frac{\Lambda^2}{P^2}$$

The Essence of Renormalization

★ It is convenient to define the kinematic (Mandelstam) variables:

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

These variables satisfy the relation:

$$s + t + u = 4\,m^2$$

Writing out the p_i's in the center-of-mass frame, we can see that s, t, and u are related to “rather mundane quantities” such as the center-of-mass energy $\mathcal{E}$ and scattering angle θ:

$$s = 4\,\mathcal{E}^2$$
$$t = -2\,|\vec{k}|^2\,(1 - \cos\theta)$$
$$t = -2\,|\vec{k}|^2\,(1 + \cos\theta)$$

where $|\vec{k}|$ is the center-of-mass momenta of the incident and scattered particles: $\mathcal{E}^2 = |\vec{k}|^2 + m^2$.

The Essence of Renormalization

★ In terms of the Mandelstam variables, the scattering amplitude $\mathcal{M}$ of 2 particles for the φ^4 theory up to order λ^2 is

$$
\begin{aligned}
i\mathcal{M} &= -i\,\lambda + f(s) + f(t) + f(u) + \mathcal{O}(\lambda^3) \\
&= -i\,\lambda + i\,\lambda^2\,\frac{1}{32\pi^2}\left(\log\frac{\Lambda^2}{s} + \log\frac{\Lambda^2}{t} + \log\frac{\Lambda^2}{u}\right) + \mathcal{O}(\lambda^3)
\end{aligned}
$$

This expression tells us the scattering amplitude in terms of the scattering parameters s, t, and u.

★ In an experiment we can measure s, t, u, and $\mathcal{M}$ but, what is the coupling constant λ? Actually, λ cannot be measure, it is a parameter of the Lagrangian. Also, what about the cutoff Λ? What is the use of this formula if we cannot measure either λ or Λ?

The Essence of Renormalization

★ We have to think more carefully. Imagine that that we have performed an experiment and found out that at some known values s_0, t_0, u_0, the scattering amplitude $\mathcal{M}$ has a value that (for the sake of the explanation) we will denote as $-\lambda_R$. If we now put these values in our formula, we obtain

$$-i\,\lambda_R = -i\,\lambda + i\,\lambda^2 \frac{1}{32\pi^2}\left(\log\frac{\Lambda^2}{s_0} + \log\frac{\Lambda^2}{t_0} + \log\frac{\Lambda^2}{u_0}\right) + \mathcal{O}(\lambda^3)$$

Now, we can use this equation to eliminate λ in favor of λ_R (the experimental value $\mathcal{M}$ at s_0, t_0, and u_0):

$$\lambda = \lambda_R + \lambda_R^2 \frac{1}{32\pi^2}\left(\log\frac{\Lambda^2}{s_0} + \log\frac{\Lambda^2}{t_0} + \log\frac{\Lambda^2}{u_0}\right) + \mathcal{O}(\lambda_R^3)$$

and the amplitude is

$$\begin{aligned} i\mathcal{M} = {} & -i\,\lambda_R - i\,\lambda_R^2 \frac{1}{32\pi^2}\left(\log\frac{\Lambda^2}{s_0} + \log\frac{\Lambda^2}{t_0} + \log\frac{\Lambda^2}{u_0}\right) \\ & + i\,\lambda_R^2 \frac{1}{32\pi^2}\left(\log\frac{\Lambda^2}{s} + \log\frac{\Lambda^2}{t} + \log\frac{\Lambda^2}{u}\right) + \mathcal{O}(\lambda_R^3) \end{aligned}$$

which can be simplified to get ...

The Essence of Renormalization

...

$$i\mathcal{M} = -i\,\lambda_R + i\,\lambda_R^2\,\frac{1}{32\pi^2}\left(\log\frac{s_0}{s} + \log\frac{t_0}{t} + \log\frac{u_0}{u}\right) + \mathcal{O}(\lambda_R^3)$$

Lo and behold!. The cutoff vanishes!! This expression gives the scattering amplitude at any values of the scattering parameters (s, t, and u) in terms of physical quantities s_0, t_0, u_0, and λ_R. Note that the experimental value $\mathcal{M}$ at s_0, t_0, and u_0, namely λ_R, plays the role of the coupling constant.

The Essence of Renormalization

★ λ_R is called the renormalized coupling constant. Actually, λ_R is not constant. If we measure the scattering amplitude at some other values s_0', t_0', and u_0', we would get a different value; let us call this value $-\lambda_R'$.

$$i\mathcal{M} = -i\,\lambda_R' + i\,\lambda_R'^2\,\frac{1}{32\pi^2}\left(\log\frac{s_0'}{s} + \log\frac{t_0'}{t} + \log\frac{u_0'}{u}\right) + \mathcal{O}(\lambda_R'^3)$$

The same formula with different values for the coupling constant and the scattering parameters used to measure it, but it gives the same $\mathcal{M}$.

The Essence of Renormalization

★ The physical coupling constant λ_R is a function of s_0, t_0, and u_0. For theoretical purposes it is much less cumbersome to set s_0, t_0, and u_0 equal to μ^2 an thus use, the simpler definition

$$-i\,\lambda_R(\mu) = -i\,\lambda + i\,\lambda^2\,\frac{3}{32\pi^2}\,\log\frac{\Lambda^2}{\mu^2} + \mathcal{O}(\lambda^3)$$

This is purely for theoretical convenience. In fact, since s_0, t_0, and u_0 have to satisfy $s_0 + t_0 + u_0 = 4m^2$, the kinematic point $s_0 = t_0 = u_0 = \mu^2$ cannot be reached experimentally. Then, the scattering amplitude can be written as

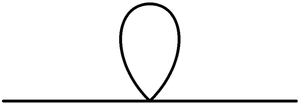
$$i\mathcal{M} = -i\,\lambda_R(\mu) + i\,\lambda_R^2(\mu)\,\frac{1}{32\pi^2}\left(\log\frac{\mu^2}{s} + \log\frac{\mu^2}{t} + \log\frac{\mu^2}{u}\right) + \mathcal{O}(\lambda_R^3(\mu))$$

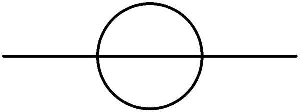
Counterterms and Renormalized Perturbation Theory

★ The inverse full φ propagator is

$$D^{-1}(p) = -i\,(p^2 - m^2) - \Gamma^{(2)}_{1\mathrm{PI}}$$

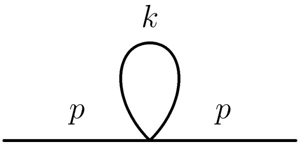
The two diagrams that contribute to $\Gamma^{(2)}_{1\mathrm{PI}}$ up to order λ^2 are





Counterterms and Renormalized Perturbation Theory

★ The first diagram:



gives

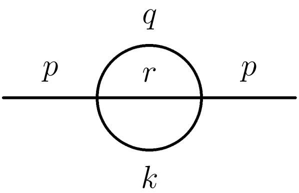
$$I_1 = \frac{1}{2}(-i\,\lambda)\int_k \frac{i}{k^2 - m^2 + i\,\epsilon}$$

We see that I_1 is independent of p and it depends quadratically on the cutoff Λ:

$$I_1 \sim \int^{\Lambda} \frac{d^4 k}{k^2} \sim \Lambda^2$$

Counterterms and Renormalized Perturbation Theory

★ The second diagram:



with $r = p + k + q$, gives

$$I_2 = \frac{1}{6}\,(-i\,\lambda)^2 \int_k \int_q \frac{i}{k^2 - m^2 + i\,\epsilon}\,\frac{i}{q^2 - m^2 + i\,\epsilon}\,\frac{i}{(p+k+q)^2 - m^2 + i\,\epsilon}$$

Counterterms and Renormalized Perturbation Theory

★ By Lorentz invariance I_2 is a function of p^2 that can be expand in powers of p^2:

$$I_2 = D + E\,p^2 + F\,p^4 + \cdots$$

★ D is obtained by taking $p = 0$ and we can see that it depends quadratically on the cutoff Λ:

$$D \sim \int^{\Lambda} \frac{d^8 K}{K^6} \sim \Lambda^2$$

★ E is obtained by differentiating I_2 with respect to p twice and setting $p = 0$. Each derivative decreases a power of k and q in the integrand and so E depends logarithmically on the cutoff Λ:

$$E \sim \int^{\Lambda} \frac{d^8 K}{K^8} \sim \log \Lambda$$

★ F is obtained similarly by differentiating I_2 with respect to p four times and setting $p = 0$. The integral

$$F \sim \int^{\Lambda} \frac{d^8 K}{K^{10}} \sim \frac{1}{\Lambda^2}$$

is convergent (we can safely take $\Lambda \to \infty$) and therefore cutoff independent. Similarly the rest of the terms $(+\cdots)$ are cutoff independent and we do not have to worry about them.

Counterterms and Renormalized Perturbation Theory

★ Then summing I_1 and I_2, the inverse propagator up to order k^2 has the form:

$$D^{-1}(p) = -i\,(p^2 - m^2 + a + b\,p^2)$$

where is a is quadratically divergent and b is logarithmically divergent. The full propagator (up to an $i\epsilon$ term)

$$D(p) = \frac{i}{(1+b)\,p^2 - (m^2 - a)} = \frac{i\,\frac{1}{1+b}}{p^2 - \frac{m^2-a}{1+b}}$$

has the pole in p^2 shifted to

$$m_R^2 \equiv \frac{m^2 - a}{1+b}$$

which we identify as the renormalized ("physical") mass. *Quantum fluctuations have shifted the mass.*

★ The pole (up to a factor i) in the full propagator is no longer 1 but

$$Z \equiv \frac{1}{1+b}$$

To understand this shift in the residue, recall that the coefficient of p^2 in the propagator is 1 because (for no better choice) we took the coefficient of $\frac{1}{2}(\partial\varphi)^2$ in the Lagrangian equal to 1. However, we have seen that quantum fluctuations have shifted this "normalization" of the field to $1/(1+b)$. For historical reasons this is known as "wave function renormalization" although there is no wave function anywhere; the modern term is field renormalization.

Counterterms and Renormalized Perturbation Theory

★ What we have been doing so far is known as *bare perturbation theory*. We may have put the subscript $_0$ on what we have been calling φ, m, and λ. The field φ_0 is known as the bare field, and m_0 and λ_0 are known as the bare mass and bare coupling respectively. Then, the Lagrangian of th φ^4 theory should have been written as

$$\mathcal{L} = \frac{1}{2}\,(\partial\varphi_0)^2 - \frac{1}{2}\,m_0^2\,\varphi_0^2 - \frac{\lambda_0}{4!}\,\varphi_0^4$$

Counterterms and Renormalized Perturbation Theory

★ In the light of our discussion it seems a little awkward to work with bare quantities all the time in order for at the end of the day exchange them for renormalized ones. Wouldnt it be better to write the theory in terms of renormalized quantities?

★ If we define the *renormalized field* φ_R, using the field renormalization constant Z, as

$$\varphi_0 \equiv Z^{1/2}\,\varphi_R$$

we get a Lagrangian for φ_R:

$$\mathcal{L} = \frac{1}{2}\,Z\,(\partial\varphi_R)^2 - \frac{1}{2}\,m_0^2\,Z\,\varphi_R^2 - \frac{\lambda_0}{4!}\,Z^2\,\varphi_R^4$$

★ If we now repeat the previous calculation for the full φ_R propagator we would get

$$\frac{i}{p^2 - (m_0^2 - a)Z}$$

whose residue is 1. In terms of $m_R^2 = (m_0^2 - a)Z$ instead of m_0, the full propagator (in the approximation we used) is

$$\frac{i}{p^2 - m_R^2}$$

and ...

Counterterms and Renormalized Perturbation Theory

... $\mathcal{L}$ becomes

$$\mathcal{L} = \frac{1}{2}\, Z\,(\partial\varphi_R)^2 - \frac{1}{2}\, m_R^2\,\varphi_R^2 - \frac{\lambda_0}{4!}\, Z^2\,\varphi_R^4 - \\ - \frac{1}{2}\,\delta_m\,\varphi_R^2$$

with $\delta_m \equiv aZ$. Note that the "trick" that makes up the mass term is nothing but writing

$$m_0^2\, Z = m_R^2 + \delta_m$$

If we do the same for the coupling term by writing

$$\lambda_0\, Z^2 = \lambda_R + \delta_\lambda$$

the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\, Z\,(\partial\varphi_R)^2 - \frac{1}{2}\, m_R^2\,\varphi_R^2 - \frac{\lambda_R}{4!}\,\varphi_R^4 - \\ - \frac{1}{2}\,\delta_m\,\varphi_R^2 - \frac{\delta_\lambda}{4!}\,\varphi_R^4$$

and if we now define

$$Z = 1 + \delta_Z$$

we get ...

Counterterms and Renormalized Perturbation Theory

...

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 + \\ + \frac{1}{2}\delta_Z(\partial\varphi)^2 - \frac{1}{2}\delta_m\varphi^2 - \frac{\delta_\lambda}{4!}\varphi^4$$

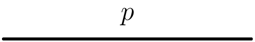
where I have drop the subscript $_R$ from the renormalized quantities.

★ Note that now the Lagrangian is written in terms of renormalized quantities at the price of having three additional terms called counterterms that have to be determined iteratively in order to renormalize the theory.

Counterterms and Renormalized Perturbation Theory

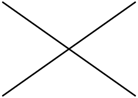
The Feynman rules for φ^4 in renormalized perturbation theory are:

- Propagator:



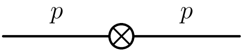
$$\frac{i}{p^2 - m^2 + i\,\epsilon}$$

- Vertex:

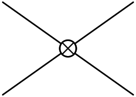


$-i\,\lambda$

- Counterterms:



$i\,(p^2\,\delta_Z - \delta_m)$



$-i\,\delta_\lambda$

Counterterms and Renormalized Perturbation Theory

★ Sometimes it is convenient to use other renormalization constants different from the counterterms.

★ The expression that relates the bare field with the renormalized field, $\varphi_0 = Z^{1/2}\varphi$, gives a relation for the Green functions of φ_0 and φ. For instance,

$$G_0^{(n)} = Z^{n/2}\, G^{(n)}$$

★ The amplitude is

$$i\mathcal{M} = \left.\frac{G^{(n)}(p_1,\ldots,p_n)}{G^{(2)}(p_1,-p_1)\cdots G^{(2)}(p_n,-p_n)}\right|_{p_a^2=m_a^2}$$

in terms of the renormalized Green functions (the field renormalization constant Z has disappeared). Therefore, we conclude that the amplitude is given by the sum of all connected, amputated diagrams for the renormalized field with on-shell external momenta.

The Renormalization Group Equation

★ We found the scattering amplitude of two particles in the φ^4 theory

$$i\mathcal{M} = -i\,\lambda(\mu) + i\,\frac{1}{32\pi^2}\,\lambda^2(\mu)\left(\log\frac{\mu^2}{s} + \log\frac{\mu^2}{t} + \log\frac{\mu^2}{u}\right) + \mathcal{O}(\lambda^3(\mu))$$

in terms of an energy scale and the renormalized coupling at such scale.

★ What is the physical meaning of $\lambda(\mu)$?

$\lambda(\mu)$ is particularly convenient for studying the physics in the regime in which the kinematic parameters s, t, and u are all of order μ^2. Then the scattering amplitude is given by $-i\lambda(\mu)$ plus small logarithmic corrections.

In contrast, if we use the coupling constant $\lambda(\mu')$ while exploring the physics in the regime with s, t, and u of order μ^2, with μ vastly different from μ', then we will have a scattering amplitude

$$i\mathcal{M} = -i\,\lambda(\mu') + i\,\frac{1}{32\pi^2}\,\lambda^2(\mu')\left(\log\frac{\mu'^2}{s} + \log\frac{\mu'^2}{t} + \log\frac{\mu'^2}{u}\right) + \mathcal{O}(\lambda^3(\mu'))$$

in which the second term (with $\log(\mu'^2/\mu^2)$ large) can be comparable to or larger than the first term. Thus, for each energy scale μ there is an appropriate coupling constant $\lambda(\mu)$.

The Renormalization Group Equation

★ Subtracting these two expressions we can easily relate $\lambda(\mu)$ and $\lambda(\mu')$ for $\mu \sim \mu'$:

$$\lambda(\mu') = \lambda(\mu) + \frac{3}{32\pi^2}\,\lambda^2(\mu)\log\frac{\mu'^2}{\mu^2} + \mathcal{O}(\lambda^3(\mu)) \tag{1}$$

We can express this as a differential "flow equation"

$$\mu\,\frac{d}{d\mu}\lambda(\mu) = \frac{3}{16\pi^2}\,\lambda^2(\mu) + \mathcal{O}(\lambda^3(\mu))$$

The description of how $\lambda(\mu)$ changes with μ is known as the renormalization group.

Note that since the constant in front of λ^2 is positive, then the coupling $\lambda(\mu)$ increases as μ increases (λ flows away from the origin) . If the constant in front of λ^2 had been negative, then the coupling $\lambda(\mu)$ would have decreased as μ increases.

★ In general, in a quantum field theory with a coupling constant g, we have the renormalization group flow equation

$$\mu\,\frac{dg}{d\mu} = \beta(g) \tag{2}$$

Outline

Lorentz Invariance: Fermions and Vectors

★ *Lorentz invariance guarantees that laws of physics are the same in all inertial frames. This the (special) relativity principle.*

So, we are interested in Lorentz invariant field theories. The Lagrangian is a scalar (no "free" Lorentz indices).

Lorentz Invariance: Fermions and Vectors

★ Fields can be classified according to the way they transform under Lorentz transformations. There are two kinds of Lorentz transformations: rotations and boosts. Spin has to do with rotations.

Classification:

$$
\begin{aligned}
(0,0) &\rightarrow \text{scalar} \\
(1/2,0) &\rightarrow \text{Weyl spinor (left, conventional)} \\
(0,1/2) &\rightarrow \text{Weyl spinor (right, conventional)} \\
(1/2,0)\oplus(0,1/2) &\rightarrow \text{Dirac spinor} \\
(1/2,0)\otimes(0,1/2)=(1/2,1/2) &\rightarrow \text{Vector field} \\
(0,1)\oplus(1,0) &\rightarrow F_{\mu\nu}
\end{aligned}
$$

★ Under a Lorentz transformation:

$$
\begin{aligned}
\text{scalar (spin 0):} &\ \varphi \rightarrow \varphi \\
\text{vector (spin 1):} &\ V^\mu \rightarrow \Lambda^\mu_\nu V^\nu \\
\text{spinor (spin 1/2):} &\ \psi \rightarrow S\psi \quad \text{with } S\gamma^\mu S^{-1} = \Lambda^\mu_\nu \gamma^\nu
\end{aligned}
$$

Fermions: Quantization

★ A free spin $1/2$ fermion of mass is described by the Lagrangian

$$\mathcal{L} = \bar{\psi}\,(i\,\gamma^{\mu}\partial_{\mu} - m)\,\psi$$

A few comments:

- ψ is a 4-component spinor.
- γ^{μ}'s are 4×4 matrices (known as Dirac's gamma matrices) that satisfy

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\,\eta^{\nu\nu}$$

  where $\eta^{\mu\nu}$ is the Minkowski metric.
- $\bar{\psi} = \psi^{\dagger}\gamma^{0}$. The reason for using $\bar{\psi}$ instead of $\psi^{\dagger}$ is that, for instance, $\bar{\psi}\psi$ is a Lorentz scalar, but $\psi^{\dagger}\psi$ is not.
- Feynman "slash" notation: $\not{a} \equiv \gamma^{\mu} a_{\mu}$.

Fermions: Quantization

★ Quantization. The generating functional of the Green functions for a free fermion of spin $1/2$ and mass m:

$$Z[\eta,\bar{\eta}] = \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\ \exp\left[\int_x \bar{\psi}\,(i\,\partial\!\!\!/ - m)\,\psi + \bar{\eta}\psi + \bar{\psi}\eta\right]$$

- Here ψ and $\bar{\psi}$ (and also the sources η and $\bar{\eta}$) are *Grassmann* functions.

  Two Grassmann numbers do not commute with each other, instead:

  $$\eta\xi = -\xi\eta$$

  This property gives rise to "curious" expressions. For instance, $\eta^2 = 0$ and the Taylor expansion of a function of a Grassmann variable is simply $f(\eta) = a + b\,\eta$.

Fermions: Quantization

- The reason why we have to use Grassmann functions instead of ordinary commuting functions in the path integral for fermions can be traced back to the spin-statistics connection.

  As we know, the Pauli exclusion principle says that bosons obey Bose-Einstein statistics and fermions obey Fermi-Dirac statistics. In short, when canonically quantizing a system (bosons or fermions), the spin-statistics connection makes the creation and destruction operators to commute or anticommute for bosons or fermions respectively.

Fermions: Quantization

★ Integrating out over ψ and $\bar{\psi}$, we get

$$\mathcal{Z}[\eta, \bar{\eta}] = \exp\left[-i \int_x \int_{x'} \bar{\eta}(x)\, S(x - x')\, \eta(x) \right]$$

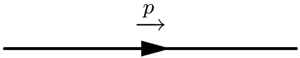
The Feynman propagator is

$$S_F(x - x') = i\, S(x - x') = \int_k \frac{i\, e^{-ik(x-x')}}{\not{k} - m + i\epsilon}$$

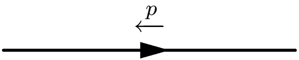
In momentum space:

$$S_F(k) = \frac{i}{\not{k} - m + i\epsilon}$$

The Feynman rule for the propagator is:



$$\frac{i}{\not{p} - m + i\,\epsilon} = \frac{i\,(\not{p} + m)}{p^2 - m^2 + i\,\epsilon}$$



$$\frac{i}{-\not{p} - m + i\,\epsilon} = \frac{-i\,(\not{p} - m)}{p^2 - m^2 + i\,\epsilon}$$

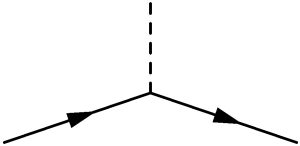
Note that the sign of p changes with the direction of the arrow.

Fermions: Quantization

★ Let us consider a simple theory with a spin 0 boson φ of mass *Mu* and a fermion ψ of mass m with Yukawa coupling $g\,\varphi\,\bar{\psi}\psi$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}\,M^2\varphi^2 + \bar{\psi}\,(i\,\partial\!\!\!/ - m)\,\psi + g\,\varphi\,\bar{\psi}\psi$$

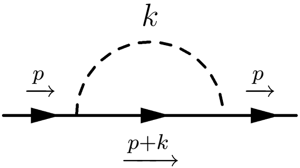
In addition to the usual Feynman rules for the propagators, we also have a Feynman rule for the vertex:



$i\,g$

Fermions: Quantization

For example we can compute the lowest order correction to the propagator (also known a self-energy) given by the diagram:



which gives

$$(ig)^2 \int_k \frac{i}{k^2 - M^2 + i\epsilon} \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2 + i\epsilon}$$

Note that this quantity is a matrix, so it cant be an amplitude. What's missing?

We need a row vector on the left and a column vector on the right. So we have to introduce a new Feynman rules that says that for an incoming fermion we write a factor $u^s(p)$ and for an outgoing fermion $\bar{u}^s(p)$:

$$\bar{u}^s(p) \left[(ig)^2 \int_k \frac{i}{k^2 - M^2 + i\epsilon} \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2 + i\epsilon} \right] u^s(p)$$

is the contribution to the amplitude.

Fermions: Quantization

★ $u^s(p)$ and $\bar{u}^s(p) = [u^s(p)]^\dagger \gamma^0$ are Dirac spinors with 4 components. For antifermions we need two spinors: $v^s(p)$ and $\bar{v}^s(p)$. Here p is the momentum of the particle and s a label for the z-component of spin. For spin $1/2$ particles, $s = +, -$ or $s = 1, 2$.

These spinor satisfy the following relations:

$$(\not{p} - m)\, u^s(p) = 0 \qquad \bar{u}^s(p)\, (\not{p} - m) = 0$$
$$(\not{p} + m)\, v^s(p) = 0 \qquad \bar{v}^s(p)\, (\not{p} + m) = 0$$

Also, with a conventional spinor normalization:

$$\bar{u}^r(p)\, u^s(p) = 2m\, \delta_{rs}$$
$$\bar{v}^r(p)\, v^s(p) = -2m\, \delta_{rs}$$
$$\sum_s u^s(p)\, \bar{u}^s(p) = \not{p} + m$$
$$\sum_s v^s(p)\, \bar{v}^s(p) = \not{p} - m$$

Quantum Electrodynamics (QED)

★ Maxwell's equation can be deduced (using the Euler-Lagrange equations) from

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and $A_\mu(x)$ is the vector potential.
Note that the field A_μ associated with the photon is a massless vector. There is no mass-term in the Lagrangian. This is consistent with experiment but from the mathematical point of view this also leads to unnecessary complications (at this stage).

So, to derive a Feynman rule for the photon we adopt a pragmatic attitude by letting the photon have a finite (however small) mass m_γ an setting $m_\gamma = 0$ at the end of the day.

We then add a photon mass term to the Lagrangian:

$$\mathcal{L}'_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_\gamma^2 A_\mu A^\mu$$

Quantum Electrodynamics (QED)

★ The generating functional is

$$\mathcal{Z}[J] = \int \mathcal{D}A_\mu \, \exp\left[i \int_x \mathcal{L}'_{\text{EM}} + A_\mu J^\mu \right]$$

with the source $J^\mu(x)$ a vector. Performing the functional integral,

$$\mathcal{Z}[J] = \exp\left[\frac{i}{2} \int_x \int_{x'} J_\mu(x)\, D^{\mu\nu}(x - x')\, J_\nu(x') \right]$$

where

$$D_{\mu\nu}(x - x') = \int_k D_{\mu\nu}(k)\, e^{ik(x - x')}$$

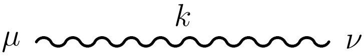
with

$$D_{\mu\nu}(k) = \frac{k_\mu k_\nu / m_\gamma^2 - \eta_{\mu\nu}}{k^2 - m_\gamma^2 + i\epsilon}$$

The Feynman propagator is

$$D^F_{\mu\nu}(k) = i\, D_{\mu\nu}(k) = \frac{i(k_\mu k_\nu / m_\gamma^2 - \eta_{\mu\nu})}{k^2 - m_\gamma^2 + i\epsilon}$$

whose Feynman rule is



$$\frac{i(k_\mu k_\nu / m_\gamma^2 - \eta_{\mu\nu})}{k^2 - m_\gamma^2 + i\epsilon}$$

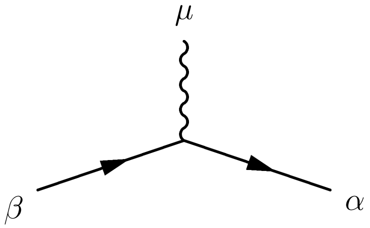
Quantum Electrodynamics (QED)

★ Now let's couple the photon to an electron with a term $e\,A_\mu\,\bar{\psi}\gamma^\mu\psi$, where e is the coupling constant. The Lagrangian

$$\mathcal{L}'_{\text{QED}} = \bar{\psi}\left[i\,\gamma^\mu(\partial_\mu - i\,e\,A_\mu) - m\right]\psi - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\,m_\gamma^2\,A_\mu A^\mu$$

describes (modulo the photon mass term) the quantum theory of electromagnetism (QED).

The Feynman rule for the vertex is



$$i\,e\,(\gamma^\mu)_{\alpha\beta}$$

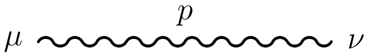
where we have explicitly showed the γ-matrix elements.

Quantum Electrodynamics (QED)

It can be showed that in actual amplitude calculations, the photon mass part in the photon propagator ($k_\mu k_\nu / m_\gamma^2$) goes away and we can set $m_\gamma = 0$.

Then, the propagator can be written as $-i\eta_{\mu\nu}/(k^2 + i\epsilon)$.

But, since we can discard the $k_\mu k_\nu / m_\gamma^2$ term, we can also add in a $k_\mu k_\nu / k^2$ term with an arbitrary coefficient. Thus, for the photon propagator we can use



$$\frac{i}{p^2 + i\epsilon}\left[(1-\xi)\frac{p_\mu p_\nu}{p^2} - \eta_{\mu\nu}\right]$$

where we can choose the number ξ to simplify our calculation as much as possible.

The choice of ξ amounts to a choice of gauge for the electromagnetic field. The choice $\xi = 1$ is known as the Feynman gauge, and the choice $\xi = 0$ is the Landau gauge. The end result must not depend on ξ.

Quantum Electrodynamics (QED)

The $p_\mu p_\nu$ term in the propagator can be obtained by adding a term

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

in the Lagrangian which is known as the gauge fixing term.

★ Summary. The QED Lagrangian:

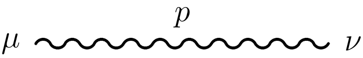
$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left[i\,\gamma^\mu(\partial_\mu - i\,e\,A_\mu) - m\right]\psi - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

External lines are amputated. For an incoming fermion line write $u^s(p)$ and for an outgoing fermion line write $\bar{u}^s(p)$. For antifermions we need $v^s(p)$ and $\bar{v}^s(p)$.

A factor (-1) has to be associated with each closed fermion line.

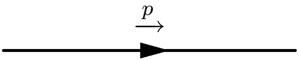
Quantum Electrodynamics (QED)

Photon Propagator:



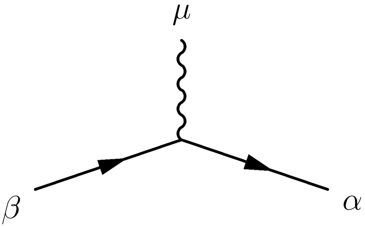
$$\frac{i}{p^2 + i\epsilon}\left[(1-\xi)\,\frac{p_\mu p_\nu}{p^2} - \eta_{\mu\nu}\right]$$

Electron Propagator:



$$\frac{i}{\not{p} - m + i\,\epsilon} = \frac{i\,(\not{p} + m)}{p^2 - m^2 + i\,\epsilon}$$

Electron Vertex:



$$i\,e\,(\gamma^\mu)_{\alpha\beta}$$

Gauge Symmetry

The QED Lagrangian can be written (dropping the gauge fixing term)

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\,(i\,\not{D} - m)\,\psi - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}$$

where we have introduced the covariant derivative

$$D_\mu \equiv \partial_\mu - i\,e\,A_\mu$$

This theory is invariant under the following transformations:

$$\psi(x) \to e^{i\,\alpha(x)}\,\psi(x) \qquad \text{(local phase transformation)}$$

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\,\partial_\mu\alpha(x)$$

This is a gauge transformation of the fields.

★ The transformation law for A_μ can be obtained by requiring the covariant derivative of ψ to transform in exactly the same way as ψ under local phase transformations.

Even more, starting from $\bar{\psi}\,(i\,\gamma^\mu\partial_\mu - m)\,\psi$ we can see that $\mathcal{L}_{\text{QED}}$ is the only way of constructing a Lagrangian invariant under local phase transformations of ψ.

Gauge Symmetry

★ Let's pose the same question for a collection of fermions: What is the field theory made out of a collection of fermions ψ_i (each with 4 components) that is invariant under local phase transformations of the fields? The answer goes as follows. Let's denote the collection of fields in a column vector as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix}$$

Then, it is possible to construct an invariant Lagrangian which has the form

$$\mathcal{L} = \bar{\psi}\,(i\,\not{D} - m)\,\psi - \frac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu}_a$$

There is an implicit sum over index a. Note that this Lagrangian looks a lot like $\mathcal{L}_{\text{QED}}$, *but*

$$D_\mu = \partial_\mu + i\,g\,A^a_\mu(x)\,t^a$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\mu A^a_\nu + g\,f^{abc}\,A^b_\mu A^c_\nu$$

with some fields $A^a_\mu(x)$ analogous to QED's A_μ, a constant g analogous to e (the different sign is conventional), some matrices t^a, and some constants f^{abc}.

Gauge Symmetry

There is a relation between the t^a's and the f^{abc}'s:

$$[t^a, t^b] = i\, f^{abc}$$

This equation tells us that the t^a's are the generators of a *Lie algebra*.

Therefore, a runs from 1 to the number of generators of the algebra. A Lie group can be constructed from the Lie algebra (the elements of the group have the form $e^{i\alpha(x)^a t^a}$).

The t^a's are realized in different representations with different dimensions; in general, the t^a's are matrices of dimension equal to the dimension of the representation. Since the t^a's act on the ψ fields; the number of components of these equals the dimension of the representation. We say that that the field is in such or such representation.

Gauge Symmetry

★ The Lagrangian

$$\mathcal{L} = \bar{\psi}\,(i\,\not{D} - m)\,\psi - \frac{1}{4}\,F^a_{\mu\nu}F_a^{\mu\nu}$$

is a non-Abelian gauge theory that describes a fermion of mass m in some representation of the Lie algebra (group) whose interaction is mediated by a gauge field $A_\mu \equiv A^a_\mu t^a$.

★ QED is an Abelian gauge theory. The generator is just 1.

★ As in QED we have to add a gauge fixing term.

Quantum Chromodynamics (QCD)

★ Quantum Chromodynamics (QCD) is the field theory that describes the strong interaction.

★ The gauge group is $SU(3)$; there are $3^2 - 1 = 8$ generators and, therefore, $a = 1, \ldots, 8$.

★ Quarks are in the fundamental representation ($\mathbf{3}$); then, $i = 1, 2, 3$ (which correspond with "colors": red, green, and blue respectively, for example). Antiquarks are the $\bar{\mathbf{3}}$ representation.

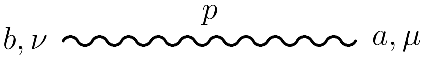
★ There are 6 flavors $(u, d), (c, s), (t, b)$.

★ The Lagrangian of QCD:

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}\, F^a_{\mu\nu} F_a^{\mu\nu} + \sum_{f=1}^{6} \bar{\psi}_f \, (i \not{D} - m_f) \, \psi_f$$

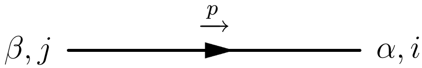
Quantum Chromodynamics (QCD)

Gluon propagator:



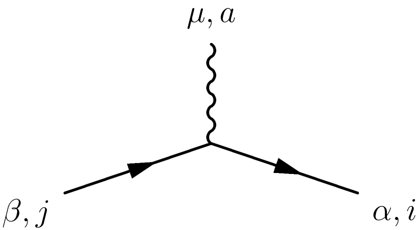
$$\frac{i\delta^{ab}}{p^2 + i\epsilon}\left[(1-\xi)\,\frac{p_\mu p_\nu}{p^2} - \eta_{\mu\nu}\right]$$

Quark propagator:



$$\frac{i\delta^{ij}(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}$$

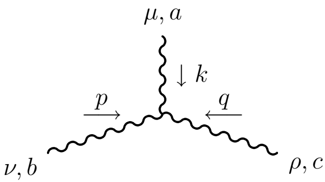
Quark-Gluon-Quark vertex:



$$i\,g\,(\gamma^\mu)_{\alpha\beta}\,(t^a)_{ij}$$

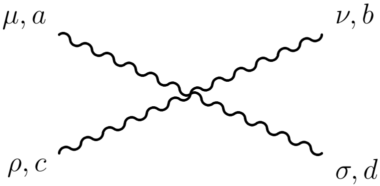
Quantum Chromodynamics (QCD)

3-Gluon vertex:



$$g f^{abc}\Big[\eta^{\mu\nu}(k-p)^{\rho}+\eta^{\nu\rho}(p-q)^{\mu}+\eta^{\rho\mu}(q-k)^{\nu}\Big]$$

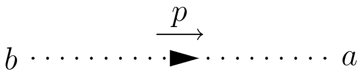
4-Gluon vertex:



$$\begin{aligned}-ig^2\big[&f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho})\\ &+f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho})\\ &+f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma})\big]\end{aligned}$$

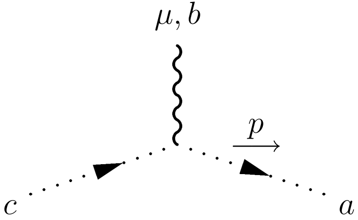
Quantum Chromodynamics (QCD)

Ghost Propagator:



$$\frac{-i\delta^{ab}}{p^2 + i\epsilon}$$

Ghost-Gluon-Ghost Vertex:



$$-g\, f^{abc}\, p^{\mu}$$

Outline

Spontaneous Symmetry Breaking

★ Weak interaction is mediated by massive vector (spin 1) bosons.

A mass term for a gauge field breaks gauge symmetry:

$$
\begin{aligned}
A_\mu^a A_a^\mu \rightarrow A_\mu^a A_a^\mu &+ \frac{2}{g} \left(\partial_\mu \alpha_a\right) A_a^\mu + \frac{1}{g^2} \left(\partial_\mu \alpha_a\right) \left(\partial^\mu \alpha_a\right) \\
&\neq A_\mu^a A_a^\mu
\end{aligned}
$$

How can we construct a gauge theory with a massive gauge field?

We will answer this question for gauge theories with a scalar that spontaneously breaks gauge symmetry.

But for the moment, let's just study a few non-gauge scalar theories.

Spontaneous Symmetry Breaking

★ A "baby" model:

$$\mathcal{L}(\varphi) = \frac{1}{2}\,(\partial_\mu\varphi)(\partial^\mu\varphi) + \frac{1}{2}\,\mu^2\,\varphi^2 - \frac{\lambda}{4}\,\varphi^4$$

This Lagrangian is symmetric under the $\varphi \to -\varphi$ transformation.

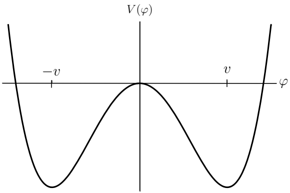
This example shows the basic idea behind generating mass terms by spontaneously breaking a symmetry.

If $\mu^2 < 0$ or the sign in front the φ^2-term was "$-$" instead of "$+$", this Lagrangian would describe a scalar particle of mass μ. But with "$+$" and $\mu^2 > 0$, the φ^2-term isn't a mass term anymore.

Spontaneous Symmetry Breaking

The Lagrangian describes a system with potential:

$$V(\varphi) = -\frac{1}{2}\,\mu^2\,\varphi^2 + \frac{\lambda}{4}\,\varphi^4$$



It has two minima at $\varphi = \pm v$ with $v = \sqrt{\mu^2/\lambda}$ and a maximum at $\varphi = 0$.

φ represents fluctuations around $\varphi = 0$, but this point is an unstable point.

The Lagrangian has to be written in terms of a field that represents fluctuations around the vacuum (a minimum)

Let's write

$$\varphi(x) = v + \eta(x)$$

Here $\eta(x)$ represents the quantum fluctuations around the minumum at $\varphi = v$. We substitute in $\mathcal{L}(\varphi)$ ...

Spontaneous Symmetry Breaking

... and arrive at

$$\mathcal{L}(\eta) = \frac{1}{2}\,(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2\,\eta^2 - \lambda\, v\, \eta^3 - \frac{\lambda}{4}\,\eta^4 + \text{const.}$$

The η^2-term now has the correct sign "$-$" for a mass term.

η represents a particle of mass $m_\eta = \sqrt{2\mu^2}$.

This way of generating mass terms is called Spontaneous Symmetry Breaking (SSB).

$\mathcal{L}(\varphi)$ shows a symmetry $\varphi \to -\varphi$ that is "hidden" in $\mathcal{L}(\eta)$.

$\mathcal{L}(\eta)$ gives the particle content.

Spontaneous Symmetry Breaking

★ A "child" model:

$$\mathcal{L}(\varphi, \varphi^*) = (\partial_\mu \varphi)^* (\partial^\mu \varphi) + \mu^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2$$

has a global phase symmetry

$$\varphi(x) \to e^{i\alpha} \varphi(x)$$

For $\mu^2 > 0$, the quadratic $\varphi^* \varphi$ has the wrong sign for a mass term.

φ is a complex field and can be written

$$\varphi = \frac{1}{\sqrt{2}} (\varphi_1 + i \varphi_2)$$

In terms of φ_1 and φ_2, the Lagrangian is

$$\mathcal{L}(\varphi_1, \varphi_2) = \frac{1}{2} (\partial_\mu \varphi_1)(\partial^\mu \varphi_1) + \frac{1}{2} (\partial_\mu \varphi_2)(\partial^\mu \varphi_2) + \frac{1}{2} \mu^2 (\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4} (\varphi_1^2 + \varphi_2^2)^2$$

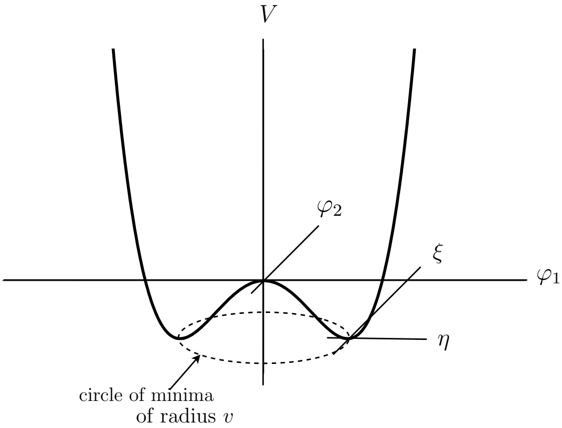
Spontaneous Symmetry Breaking

The potential

$$V(\varphi_1, \varphi_2) = -\frac{1}{2}\,\mu^2\,(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4}\,(\varphi_1^2 + \varphi_2^2)^2$$

has a circle of minima in the (φ_1, φ_2) plane of radius v such that

$$\varphi_1^2 + \varphi_2^2 = v^2 \qquad \text{with } v^2 = \mu^2/\lambda$$



Spontaneous Symmetry Breaking

We want to write the Lagrangian in terms of fields that represent fluctuations around a minimum (the vacuum). We can choose any point in the minima circle, for simplicity we take

$$(\varphi_1, \varphi_2) = (v, 0)$$

by writing

$$\varphi(x) = \frac{1}{\sqrt{2}}\left[v + \eta(x) + i\,\xi(x)\right]$$

We obtain the Lagrangian in terms of η and ξ:

$$\begin{aligned}\mathcal{L}(\eta, \xi) = \frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi) + \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2 \eta^2 \\ + \text{cubic and quartic terms in } \eta \text{ and } \xi \\ + \text{const.}\end{aligned}$$

η represents a particle of mass $m_\eta = \sqrt{2\mu^2}$,

ξ is a massless scalar which is known as a Goldstone boson.

Spontaneous Symmetry Breaking

★ In a similar way, it is easy to see that by spontaneous symmetry breaking of the Lagrangian, for N scalar fields φ_a (with $a = 1, \ldots, N$),

$$\mathcal{L}(\varphi_a) = \frac{1}{2}\,(\partial_\mu \varphi_a)(\partial^\mu \varphi_a) + \frac{1}{2}\,\mu^2\,\varphi_a\varphi_a - \frac{\lambda}{4}\,(\varphi_a\varphi_a)^2$$

we get (with $i = 1, \ldots, N-1$)

$$\begin{aligned}\mathcal{L}(\eta, \xi_i) = \frac{1}{2}\,(\partial_\mu \xi_i)(\partial^\mu \xi_i) + \frac{1}{2}\,(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2\,\eta^2 \\ + \text{const.} \\ + \text{cubic and quartic terms in } \eta \text{ and } \xi_i\end{aligned}$$

which describes a scalar η of mass $m_\eta = \sqrt{2\mu^2}$ and $N-1$ massless Goldstone bosons.

The Higgs Mechanism

★ Let's consider the a $U(1)$ gauge theory described by the Lagrangian

$$\mathcal{L}(\varphi, \varphi^*, A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^* (D^\mu \varphi) + \mu^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2$$

where, as usually, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu - i\, g\, A_\mu$. It is invariant under $U(1)$ (Abelian) gauge transformations:

$$\varphi(x) \to e^{i\alpha(x)} \varphi(x)$$

$$A_\mu(x) \to A_\mu(x) + \frac{1}{g} \partial_\mu \alpha(x)$$

As in the last section theories, the quadratic term (with $\mu^2 > 0$) has the wrong sign for a mass term.

Defining $\varphi = (\varphi_1 + i\, \varphi_2)/\sqrt{2}$, the potential $V = -\mu^2 \varphi^* \varphi + \lambda (\varphi^* \varphi)^2$ has a circle of minima in the (φ_1, φ_2) plane of radius v such that $\varphi_1^2 + \varphi_2^2 = v^2$ with $v^2 = \mu^2/\lambda$. Now, expanding the Lagrangian around $(\varphi_1, \varphi_2) = (v, 0)$ with

$$\varphi(x) = \frac{1}{\sqrt{2}} \left[v + \eta(x) + i\, \xi(x) \right]$$

we get ...

The Higgs Mechanism

...

$$
\begin{aligned}
\mathcal{L}(\eta, \xi, A_\mu) = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu \xi)(\partial^\mu \xi) + \frac{1}{2} (\partial_\mu \eta)(\partial^\mu \eta) \\
& - \mu^2 \eta^2 + \frac{1}{2} g^2 v^2 A_\mu A^\mu - g v A_\mu (\partial^\mu \xi) \\
& + \text{cubic and quartic terms in } \eta \text{ and } \xi \\
& + \text{const.}
\end{aligned}
$$

We obtain a scalar η with mass $m_\eta = \sqrt{2\mu^2}$.

A massless Goldstone boson.

A mass term for the gauge field which gets a mass $m_A = g\, v$.

The gauge symmetry which is apparent in $\mathcal{L}(\varphi, \varphi^*, A_\mu)$ is hidden in $\mathcal{L}(\eta, \xi, A_\mu)$ but ...

$\mathcal{L}(\eta, \xi, A_\mu)$ gives the particle content.

The Higgs Mechanism

★ BUT, something must be wrong because $\mathcal{L}(\varphi, \varphi^*, A_\mu)$ has 4 degrees of freedom and $\mathcal{L}(\eta, \xi, A_\mu)$ has 5.

Alternatively, we can see

$$\varphi(x) = \frac{1}{\sqrt{2}} \left[v + \eta(x) + i\,\xi(x) \right]$$

as an infinitesimal gauge transformation of a field $(v+\eta)/\sqrt{2}$ with gauge parameter $\xi/(v+\eta)$:

$$\varphi(x) = \left[1 + i \left(\frac{\xi}{v+\eta} \right) \right] \left(\frac{v+\eta}{\sqrt{2}} \right)$$

If we now write the gauge field as a gauge transformed field

$$A_\mu(x) = B_\mu(x) + \frac{1}{g}\,\partial_\mu \left(\frac{\xi}{v+\eta} \right)$$

field ξ doesn't appear in the Lagrangian because it is gauge invariant.

The Higgs Mechanism

The Goldstone boson has been eaten up by the gauge field: ξ is gone and B_μ becomes massive and gets one degree of freedom more.

We implicitly choose a gauge: the unitary gauge.

This way of getting massive gauge fields is known as the Higgs mechanism.

φ is the Higgs particle.